Problem Set 6

The following exercises are open-ended and sometimes under-specified, sometimes intentionally so. You should always feel free to ask for help from the mentors (as well as from your fellow students). **Read all the exercises before beginning to work.** You should spend at least 1 hour thinking about at least one of the starred problems.

Fixed notation: for a field K, let $\mathcal{O} = K[[x]]$ with maximal ideal $\mathfrak{m} = (x)$ and fraction field $F = K((x))$.

1. **Why can't you just be normal???** Let T be the diagonal subgroup in GL_n. Compute the normalizer $N(T) = \{g \in \mathrm{GL}_n : gTg^{-1} = T\}$. To be more specific, describe $N(T)/T$. (Hint: go looking for matrices which send T to itself but are not themselves diagonal. Find enough that you can guess the answer.)

2. **Woooord up!** The symmetric group S_n is generated by adjacent transpositions (i.e. transpositions of the form $s_i = (i\ i+1)$ for $1 \le i \le n-1$). Define the *length* $\ell(w)$ of a permutation $w \in S_n$ to be the length of the shortest presentation of w as a word in the adjacent transpositions.

   (a) Show that $\ell(w)$ is well defined.

   (b) Compute the lengths of every element of S_3. It may be helpful to compute which relations are satisfied by the s_i (Hint: $s_i^2 = 1$, but there is one additional series of relations).

   (c) What does the previous part tell you about $B\backslash \mathrm{GL}_3 / B$?

3. **Haaaaar haar, I'm a pirate!** Hopefully you have now figured out a way to equip GL_n with a nice multiplication-invariant measure. Using as building blocks the additive measure dx on F and the multiplicative measure $dx^\times$ on $F^\times$, we would like to build Haar measures on some matrix subgroups of GL_n. (That is, your answer should probably look something like $f(a,b,c,...)da\, db^\times \cdots$ where $a, b, c, \ldots$ are entries of the matrix.)

   (a) Find an invariant measure to put on $T(F)$, the subgroup of (Hint: T is isomorphic to the product $F^\times \times \cdots \times F^\times$).

   (b) Find a measure on B which is invariant under left multiplication by B. Is it also invariant under the right multiplication?

4. **Let your freak flag fly.** Recall that in GL_2 we have explicitly described the flag variety GL_2 / B: every coset can be represented by either

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ 1 & d \end{pmatrix}.$$

   So we can identify the flag variety with $K \cup \{\infty\}$ in this case. What are the left B-orbits? (There should be 2, corresponding to the 2 $B \times B$ orbits in GL_2.)

5. **Convolutations!!** Let $K = \mathbb{F}_q$. Working in GL_2, take the indicator functions

$$1_e \quad \text{and} \quad 1_{(1\ 2)}$$

of the two $B \times B$-orbits. Can you compute the convolution $(1_e \star 1_{(1\ 2)})$? (Just the basic convolution formula in G, no weird $B\backslash G/B$ stuff here.) Since you are working over $\mathbb{F}_q$, everything involved is just a finite sum. The result will be another $B \times B$-invariant function, so it can be expressed as a linear combination of 1_e and $1_{(1\ 2)}$. What are the coefficients?

6. **Eternal Sunshine of the Symplectic Mind.** Let G denote the group of 4x4 matrices A such that

$$A^T J A = J, \quad \text{where} \quad J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Let $B \leq G$ denote the subgroup of matrices which are upper triangular and contained in G. We want to study B-row and column operations in this group. That is, we want to study the left and right actions of B on G.

   (a) Can you figure out what the "elementary" row or column operations should be in G? Which matrices do they correspond to? Using your list, can you informally determine the dimension of G? What about of B?

   (b) Can you characterize the $B \times B$ orbits in G? Can you guess at orbit closures?

7. **More convoluted than expected...** Let G be a group, with H a subgroup.

   (a) Consider the diagonal action of G on $H\backslash G \times G/H$. Show that the quotient of this space by G can be identified with $H\backslash G/H$. (Feel free to work in whatever context you are comfortable with – the important thing is having the right formula for a bijection of sets).

   (b) Suppose G acts on X. Show that $\mathrm{Fun}(X/G)$ can naturally be identified with G-invariant functions on X.

   (c) Recall that $\mathrm{Fun}(X \times X)$ is always a ring, with operation defined by

$$(f_1 \star f_2)(x, y) = \sum_{z \in X} f_1(x, z) f_2(z, y).$$

   Now identify a ring structure on $\mathrm{Fun}(H\backslash G/H)$ using the previous parts. If f_1, f_2 are two $H \times H$-invariant functions on G, what is their convolution product?