Problem Set 5

The following exercises are open-ended and sometimes under-specified, sometimes intentionally so. You should always feel free to ask for help from the mentors (as well as from your fellow students). **Read all the exercises before beginning to work.** You should spend at least 1 hour thinking about at least one of the starred problems.

Fixed notation: for a field K, let $\mathcal{O} = K[[x]]$ with maximal ideal $\mathfrak{m} = (x)$ and fraction field $F = K((x))$.

1. **Fun fun fun til your daddy takes the T-bird away.**[1]

   (a) Let X be a finite set. Show that every linear map from $\mathrm{Fun}(X)$ to itself can be "represented by a distributional kernel", i.e. some element of $\mathrm{Fun}(X \times X)$. (Hint: if $\mathbb{R}^n = \mathrm{Fun}(\{0, ..., n-1\})$, what is matrix multiplication?)

   (b) Suppose X is a finite set equipped with an action by a finite group G. Characterize the elements of $\mathrm{Fun}(X \times X)$ which represent linear maps $\mathrm{Fun}(X) \to \mathrm{Fun}(X)$ which also commute with the G-action. (Hint: try to get an answer of the form $\mathrm{Fun}(Y)$ for some set Y).

2. **What the Hecke??**

   (a) Let's consider $\mathrm{Fun}(G/H)$, where H is some subgroup of our finite group G. Explain how to equip $\mathrm{Fun}(G/H)$ with the structure of a $\mathrm{Fun}(G)$-module, where the product on $\mathrm{Fun}(G)$ is convolution. (Sanity check: compare with $G = \mathbb{Z}/n\mathbb{Z}$ and $H = 1$, which is the situation you have already studied very carefully).

   (b) Does the $\mathrm{Fun}(G)$-action on $\mathrm{Fun}(G/H)$ descend to a product on $\mathrm{Fun}(G/H)$?

   (c) Give a nice characterization of the endomorphism algebra of $\mathrm{Fun}(G/H)$ as a $\mathrm{Fun}(G)$-module. (This algebra $\mathrm{End}_G(\mathrm{Fun}(G/H))$ is called a *Hecke algebra*).

   (d) For $G = \mathrm{GL}_n(\mathbb{F}_q)$, and B the subgroup of upper triangular matrices, what is your answer to the previous part? How is it related to what we've already studied? Can you say anything concrete about this algebra (esp. for $G = \mathrm{GL}_2$).

3. **Weightlifting.** Let T be the subgroup of GL_n consisting of diagonal matrices. This group acts on $\mathrm{Mat}_{n\times n}$ by *conjugation*, which is a linear action for each $t \in T$. Since the group is commutative and the action is diagonalizable, there exists a simultaneous eigenbasis for $\mathrm{Mat}_{n\times n}(k) \cong k^{n^2}$ for all the matrices in T. Each eigenspace is now associated to a *function* $T \to k^\times$: for each simultaneous eigenvector, we get a collection of its eigenvalues, one for each $t \in T$. Describe the eigenfunctions and their multiplicities for this action.

4. **Do it again, but not abelian.** Since we have additive measures on $\mathcal{O}$, F, we can take the product measure to get induced additive measures on $\mathrm{Mat}_{n\times n}(\mathcal{O}) \cong \mathcal{O}^{n^2}$ and $\mathrm{Mat}_{n\times n}(F) \cong F^{n^2}$. For notation, we could call our matrix entries

$$A = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$$

and denote the corresponding additive measures $dwdxdydz$.

[1] https://www.youtube.com/watch?v=MHuIEl2-lzs

(a) Extending our integration theory to this additive matrix group is easy. For example, consider the function $F : \mathrm{Mat}_{n\times n}(F) \to \mathbb{C}$ defined by

$$F(A) = \begin{cases} 1 & \text{if } w \in \mathfrak{m} \text{ OR } z \in \mathfrak{m}^2 \\ 0 & \text{otherwise.} \end{cases}$$

Compute $\int F dw dx dy dz$.

(b) What is the volume of $\mathrm{GL}_n(\mathcal{O}) \subset \mathrm{Mat}_{n\times n}(\mathcal{O})$ with respect to the additive measure?

(c) How does left multiplying $\mathrm{Mat}_{n\times n}$ by an invertible matrix $A \in \mathrm{GL}_n$ scale volume in the additive measure? Try some examples, maybe specialize to $n = 2$.

(d) Using the above, can you find a way to change the additive measure on GL_n to make it invariant under multiplication? Your formula should generalize the computation $dx^\times = dx/|x|$ in the case $n = 1$.

5. **Get your thoughts in order.** Recall the matrix identity from class

$$\begin{pmatrix} \frac{n}{1-n^2} & \frac{-n^2}{1-n^2} \\ 0 & n \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which we used to see that the orbit corresponding to the identity permutation is in the closure of the orbit corresponding to $\sigma = (1\ 2)$. Possibly using the algorithm Jared proposed, try to generalize this matrix identity to GL_n.

We had an imprecise idea today that this should correspond to a partial order on S_n where permutations with more elements in inverted order are bigger. Can you make a precise definition?

6. **Affine day to die.** Consider the left and right actions of $\mathrm{GL}_2(\mathcal{O})$ on $\mathrm{GL}_2(F)$ by "nonnegative valuation" row and column operations. Can you find a canonical form/characterize the orbits $\mathrm{GL}_2(\mathcal{O})\backslash \mathrm{GL}_2(F)/\mathrm{GL}_2(\mathcal{O})$?

(Harder, in case you get bored over the weekend) Define

$$I = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{m} & \mathcal{O} \end{pmatrix} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, d \in \mathcal{O}, c \in \mathfrak{m} \right\}.$$

Prove that this is a subgroup of $\mathrm{GL}_2(\mathcal{O})$. Note that this looks like some perverse hybrid of B and GL_n: if we view these matrices in the residue field $\mathcal{O}/\mathfrak{m}$, they become upper triangular, but in the fraction field they just look like any old matrices. Can you say anything about $I\backslash \mathrm{GL}_2(F)/\mathrm{GL}_2(\mathcal{O})$?

7. **Have you touched grass today?** Define the Grassmannian $G_{k,n}$ as the set of k-dimensional subspaces in an n-dimensional vector space V over a field K. If $K = \mathbb{R}$ or $\mathbb{C}$, this is a nice manifold, and it is a well-behaved algebrao-geometric object over any field. Its geometry and topology are interesting. Find a way to describe the Grassmannian as the quotient of $\mathrm{GL}_n(K)$ by some subgroup H, similar to B. (This is one of many ways to give it a topology). Interpret your result in terms of canonical forms for a restricted kind of row operations,

Lets also take this connection in the other direction. Does the space of orbits GL_n/B parameterize some kind of linear-algebraic data, the same way that the Grassmannian GL_n/H parameterizes subspaces?