Problem Set 4

The following exercises are open-ended and sometimes under-specified, sometimes intentionally so. You should always feel free to ask for help from the mentors (as well as from your fellow students). **Read all the exercises before beginning to work.** You should spend at least 1 hour thinking about at least one of the starred problems.

Fixed notation: for a field K, let $\mathcal{O} = K[[x]]$ with maximal ideal $\mathfrak{m} = (x)$ and fraction field $F = K((x))$.

1. **Cold Case?** (Ongoing) You now understand that $B \times B$ orbits (row AND column operations) on GL_n are parameterized by elements of the symmetric group S_n. Denote by X_σ the orbit corresponding to $\sigma \in S_n$. Which orbits are contained in the closure of other orbits? In other words, for which $\tau \in S_n$ can you find a convergent sequence of elements in X_σ whose limit is in X_τ?

2. **Multiplication station.** Let $K = \mathbb{F}_q$. We understand well now the "Haar" measure on $(\mathcal{O}, +)$, that is, the measure which is invariant under *additive* shifts. In this exercise, we will build the Haar measure for the simplest possible matrix groups, $\mathrm{GL}_1(\mathcal{O}) = \mathcal{O}^\times$ and $\mathrm{GL}_1(F) = F^\times$. We want to define measures (that is, assign volumes to nice subsets) on $\mathcal{O}^\times$ and $F^\times$ which are invariant under *multiplicative* shifts.

   (a) Even though $\mathcal{O}^\times$ is an open (and closed) subset of $\mathcal{O}$, we can't just restrict the same additive measure to $\mathcal{O}^\times$. Why not?

   (b) We observed this morning that we can think of the $U^{(i)}$ as a nice family of concentric disks around 1 in $\mathcal{O}^\times$. Check that defining open sets to be unions of (multiplicative) translates of these gives a topology. If we normalize $\mathcal{O}^\times$ to have volume 1, what should the volumes of the $U^{(i)}$ be?

   (c) Describe $F^\times/\mathcal{O}^\times$. What topology should we put on this quotient? Draw a picture of $F^\times$, and describe the topology which comes from extending the one on $\mathcal{O}^\times$.

   (d) Remembering that we want multiplication to preserve the volume, describe how to assign volumes to nice open subsets of $F^\times$.

   (e) (Less important) If we called the original additive measure dx, can you give a formula for the "density" of this new measure $dx^\times$ in terms of dx and x?

3. **Affine day to die.** Consider the left and right actions of $\mathrm{GL}_2(\mathcal{O})$ on $\mathrm{GL}_2(F)$ by "nonnegative valuation" row and column operations. Can you find a canonical form/characterize the orbits $\mathrm{GL}_2(\mathcal{O})\backslash \mathrm{GL}_2(F)/\mathrm{GL}_2(\mathcal{O})$?

   (Harder, in case you get bored over the weekend) Define

$$I = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{m} & \mathcal{O} \end{pmatrix} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, d \in \mathcal{O}, c \in \mathfrak{m} \right\}.$$

   Prove that this is a subgroup of $\mathrm{GL}_2(\mathcal{O})$. Note that this looks like some perverse hybrid of B and GL_n: if we view these matrices in the residue field $\mathcal{O}/\mathfrak{m}$, they become upper triangular, but in the fraction field they just look like any old matrices. Can you say anything about $I\backslash \mathrm{GL}_2(F)/\mathrm{GL}_2(\mathcal{O})$?

4. **Have you touched grass today?** Define the Grassmannian $G_{k,n}$ as the set of k-dimensional subspaces in an n-dimensional vector space V over a field K. If $K = \mathbb{R}$ or $\mathbb{C}$, this is a nice manifold, and it is a well-behaved algebrao-geometric object over any field. Its geometry and topology are interesting. Find a way to describe the Grassmannian as the quotient of $\mathrm{GL}_n(K)$ by some subgroup H, similar to B. (This is one of many ways to give it a topology). Interpret your result in terms of canonical forms for a restricted kind of row operations,

   Lets also take this connection in the other direction. Does the space of orbits GL_n / B parameterize some kind of linear-algebraic data, the same way that the Grassmannian GL_n / H parameterizes subspaces?

5. **Stabilizers to Full!** This morning, you showed an example of computing the orbit-dimension for a particular RREF form. Can you generalize this computation to arbitrary RREF forms? Can you say anything about the closure relations?