Consuman REU Group **June [. . .], 2025**

Problem Set 3

The following exercises are open-ended and sometimes intentionally under-specified. You should always feel free to ask for help from the mentors (as well as from your fellow students). **Read all the exercises before beginning to work.** You should spend at least 1 hour thinking about at least one of the starred problems.

1. ***Maximum Power!** What are the maximal ideals in $K[[x]]$? Hint: an element is a unit if and only if it is not contained in any maximal ideal. What can you say about the group structure of $K[[x]]$? Can you rephrase any of this in terms of the function v defined in the previous problem set?

2. ***You say you want a convolution, well, you know. We all wanna change the world.** Recall that we discovered two natural collections of functions $\mathrm{Fun}(\mathbb{Z}/n\mathbb{Z}, k)$:

   (a) The "standard" basis, consisting of indicator functions $1_x, x \in \mathbb{Z}/n\mathbb{Z}$, and

   (b) The "character" basis, consisting of vectors of the form

   $$v_\lambda = \begin{pmatrix} 1 \\ \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{n-1} \end{pmatrix}$$

   where λ is an nth root of unity. (For this exercise, let's assume $k = \mathbb{C}$.)

   Let's study the relationship between these bases.

   (a) Show that the character basis is indeed a basis for $\mathrm{Fun}(\mathbb{Z}/n\mathbb{Z}, \mathbb{C})$, first when $n = 3$ and then in general.

   (b) Prove that every homomorphism $\mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^*$ is of the form $x \mapsto \exp(2\pi i x/n)$. (A *character* of a group G is such a homomorphism).

   (c) Check again that the elements of the character basis are indeed characters of $\mathbb{Z}/n\mathbb{Z}$, this time without T vs. T^{-1} problems.

   (d) Compute the change of basis matrix between the standard and character bases.

   (e) Let's write $\widehat{f}(\lambda)$ for the unique scalar such that $f = \sum_\lambda \widehat{f}(\lambda) v_\lambda$. Show that $\widehat{f_1 \star f_2}(\lambda) = \widehat{f_1}(\lambda)\widehat{f_2}(\lambda)$. That is, the transform $f \mapsto \widehat{f}$ sends convolution to point-wise multiplication.

3. ***Taking up space.** Let $K = \mathbb{F}_q$ be a finite field. Our goal is to define volumes for certain well-behaved subsets of $K[[x]]$.

   (a) Let's say we make the volume of the whole space 1. What is the volume of the subset $\{f : f(0) = 0\}$? To what other subsets can you ascribe a volume using the same logic?

   (b) For each power series f, multiplication by f defines an operator on $K[[x]]$. How much does multiplication by f scale volumes, as a function of f? (For example, how much does multiplication by x^k change the volume of the whole space?).

   (c) As usual, the jump from volumes to integration is not so hard. Consider the function $\phi : f \mapsto f(0)$. Integrate this function over the power series ring $K[[x]]$.

4. **(╯°□°)╯︵ ┻━┻.** Describe the fraction field $K((x))$ of $K[[x]]$. Extend the function v to $K((x))$. Bonus: can you characterize which elements of $K((x))$ are rational functions (i.e. can be written as f/g for polynomials f, g?)

5. **Cold Case?** (Ongoing) You now understand that $B \times B$ orbits (row AND column operations) on GL_n are parameterized by elements of the symmetric group S_n. Denote by X_σ the orbit corresponding to $\sigma \in S_n$. Which orbits are contained in the closure of other orbits? In other words, for which $\tau \in S_n$ can you find a convergent sequence of elements in X_σ whose limit is in X_τ?

6. **Stabilizers to Full!** This morning, you showed an example of computing the orbit-dimension for a particular RREF form. Can you generalize this computation to arbitrary RREF forms? Can you say anything about the closure relations?