Problem Set 2

The following exercises are open-ended and sometimes intentionally under-specified. You should always feel free to ask for help from the mentors (as well as from your fellow students). **Read all the exercises before beginning to work.** You should spend at least 1 hour thinking about at least one of the starred problems.

1. ***Unlimited Power!** Let F be a field, and define the *ring of formal power series* $F[[t]]$ to be the ring whose elements are power series in one variable with coefficients in K:

$$\sum_{i=0}^{\infty} a_i t^i.$$

We call these "formal" power series because we demand no kind of convergence condition. In practice, for most fields F other than $\mathbb{R}$ and $\mathbb{C}$, there is no meaningful notion of convergence of power series.

(a) Define ring operations, a unit, etc on $F[[t]]$. These should be compatible with the usual ring structure on $F[t] \subset F[[t]]$.

(b) Let R be the convolution ring of F-valued sequences $(a_i)_{i\in\mathbb{N}}$. That is, the addition is pointwise, and the multiplication is defined by

$$((a_i) \star (b_j))_k = \sum_{i+j=k} a_i b_j.$$

Show that R is isomorphic to $F[[t]]$. (For our analysts: make the formal substitution $t = e^{2\pi i x}$, and this isomorphism becomes a certain fact about Fourier series. What is it?)

(c) Define the function v on $f \in F[[t]]$ by

$$v(p) := \min\{i : a_i \neq 0\} \in \mathbb{N}.$$

Show that $v(p)$ can be used to define a metric on $F[[t]]$, and that it is complete with respect to this metric.

2. ***If you build it, they will take p-th powers.** Let $\mathbb{F}_q$ the finite field with q elements, $q = p^e$. Our friend Rob is a freshman in college and studying the following operator on $\mathbb{F}_q$:

$$\mathrm{Rob}_p : a \mapsto a^p$$

(a) Prove (to Rob's surprise) that Rob_p from $\mathbb{F}_q$ to itself is a ring homomorphism.

(b) What can you say about the elements in $\mathbb{F}_q$ which map to 0 under Rob_p? What can you say about the elements which map to 1?

(c) Can you describe which elements map to themselves under Rob_p? What about under $\mathrm{Rob}_q = (\mathrm{Rob}_p)^e$ (composing Rob_p with itself e times)? Try some examples (and not just $\mathbb{Z}/p$ examples!), make conjectures, try to prove them.

3. ***How convoluted...** Let k be a field. Consider the n-dimensional vector space k^n. Consider the linear operator T on k^n which acts on column vectors in the following way:

$$\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix} \mapsto \begin{pmatrix} a_{n-1} \\ a_0 \\ a_1 \\ \vdots \\ a_{n-2} \end{pmatrix}.$$

I.e. every entry is shifted down by one index, except for the final entry, which moves to the top.

(a) What can you say about this operator T? (e.g. Can you write a matrix representation? What are its eigenvalues? Is it diagonalizable?)

(b) The vector space of k-valued functions $\mathrm{Fun}(\mathbb{Z}/n\mathbb{Z}, k)$ is also n-dimensional. Interpret your answers to part (a) in terms of functions on $\mathbb{Z}/n\mathbb{Z}$.

(c) The group $\mathbb{Z}/n\mathbb{Z}$ acts on itself by addition. This induces an action of $\mathbb{Z}/n\mathbb{Z}$ on $\mathrm{Fun}(\mathbb{Z}/n\mathbb{Z}, k)$. Understand what that sentence means. What can you say about the action of each element of $\mathbb{Z}/n\mathbb{Z}$?

(d) Extend the $\mathbb{Z}/n\mathbb{Z}$ action on $\mathrm{Fun}(\mathbb{Z}/n\mathbb{Z}, k)$ to an action of $\mathrm{Fun}(\mathbb{Z}/n\mathbb{Z}, k)$ on itself. Can you do so in a way that makes $\mathrm{Fun}(\mathbb{Z}/n\mathbb{Z}, k)$ into a k-algebra?

(Hint: write 1_x for the indicator function of a singleton $\{x\} \subseteq \mathbb{Z}/n\mathbb{Z}$. It would be nice to demand that the action of 1_x coincides with the action of x that you already constructed.)

4. **Case closed.** Prove that all the left B-orbits in GL_2 (GL_n?) are closed.

5. x **marks the spot.** Jared described a process for constructing a matrix with any arbitrary characteristic polynomial. Prove that these "Jared matrices" do indeed have the desired characteristic polynomial. What is their minimal polynomial?

We made the funny observation that matrices act on vector spaces, and that $\mathbb{F}_8 = \mathbb{F}_2[x]/(x^3 + x^2 + 1)$ is an $\mathbb{F}_2$-vector space. Can you describe a linear operator on $\mathbb{F}_8$ which has x^3+x^2+1 as its characteristic polynomial? Describe a similarly tautological way to interpret Jared's construction for any polynomial.

6. **What degeneracy!** Consider again the left multiplication action of GL_n on $\mathrm{Mat}_{n\times n}$. We have proven that the orbit consisting of all invertible matrices is dense, in particular it should be "n^2-dimensional," whatever that means.

If I give you a different GL_n-orbit (say, by giving you a non-identify RREF form), can you predict the dimension of that orbit? (Hint: you should think about dimension informally as "number of free variables." Something something orbit-stabilizer?!)

7. **Order in the court!** (Challenging, for you to think about) You now understand that $B \times B$ orbits (row AND column operations) on GL_n are parameterized by elements of the symmetric group S_n. Denote by X_σ the orbit corresponding to $\sigma \in S_n$. Which orbits are contained in the closure of other orbits? In other words, for which $\tau \in S_n$ can you find a convergent sequence of elements in X_σ whose limit is in X_τ?