References

- Cho, Sungmun and Lee, Yuchan. "An explicit formula for the orbital integrals on the spherical Hecke algebra of GL_3," arXiv preprint. This is the prototypical computation that we are trying to imitate, extracted in a no-frills paper.
- Cho, Sungmun, et al. "Stable orbital integrals for classical Lie algebras and smooth integral models," arXiv preprint. This larger paper extends the strategy for counting stable orbital integrals to give some partial results and recurrences for $\mathrm{GL}_n, U_n, \mathrm{Sp}_{2n}$. The main places that I see for potential improvement can be found in section 1.4, where they explain why they are restricting to the limited collection of groups that they do.
- Yun, Zhiwei. "Lectures on Springer theories and orbital integrals," IAS/Park City Mathematics Series. Sections 2 and 3 of these notes are the default reference for orbital integrals and affine Springer fibers (the geometric objects parameterizing the orbits). It contains a lot of geometry which is more advanced than you need, but the examples are great.
- Kottwitz, Robert. "Orbital Integrals on GL_3," American Journal of Mathematics, Vol. 102, No. 2 (Apr., 1980), pp. 327-384. This is a completely different approach to computing orbital integrals, using Bruhat-Tits theory. Feel free to glance at it to get a sense of the flavor.

Exercises

Fixed notation: for a field K, let $\mathcal{O} = K[[t]]$ with maximal ideal $\mathfrak{m} = (t)$ and fraction field $F = K((t))$. Write v for the additive valuation function on F.

1. Generalize the regular diagonalizable equi-valuation orbital integral computation from GL_2 to GL_n. That is, compute the orbital integral $O_\gamma(\mathbf{1}_{\mathfrak{g}(\mathcal{O})})$ "by hand" for matrices of the form

$$\gamma = \begin{pmatrix} a_1 t^k & & \\ & \ddots & \\ & & a_n t^k \end{pmatrix}$$

   where the a_i are elements of $\mathcal{O}^\times$ with distinct leading terms. What can you say if you relax the requirements that the valuations are equal?

2. Let $G = \mathrm{GL}_n$. Recall the Iwasawa decomposition $G(F) = B(F)G(\mathcal{O})$ and the Levi decomposition $B(F) = T(F)N(F)$, where N is a normal subgroup, the "unipotent radical" of B. Beginning from these, show that we have a canonical decomposition

$$G(F) = \bigsqcup_{\mathbb{Z}^n} t^\lambda N(F) G(\mathcal{O}),$$

   where $t^\lambda \in T(F)$ denotes the matrix with t^{λ_i} along the diagonal. Use this to show that for any γ diagonal with distinct entries, the orbital integral $O_\gamma(\mathbf{1}_{\mathfrak{g}(\mathcal{O})})$ can be reduced to an integral over just $N(F)$ (Try to do this using integral techniques, but if you find it too hard, revisit this after doing the problems below involving lattices).

3. Let $\gamma \in \mathfrak{gl}_n(F) = \mathrm{Mat}_{n\times n}(F)$.

   (a) Prove that the centralizer of γ is regular semisimple (that is, there exists an extension L/F such that the centralizer $G_\gamma(L)$ is isomorphic to $(L^\times)^n$) if and only if the characteristic polynomial has distinct roots in any field in which it fully factors.

   (b) Assume γ is regular semisimple. For any partition $n_1 + \cdots + n_k = n$, let $M_{(n_i)_i} = \prod_i \mathrm{GL}_{n_i}$ be the corresponding block-diagonal subgroup (we call these Levi subgroups, by the way). Prove that γ is conjugate by $\mathrm{GL}_n(F)$ to a matrix in $M_{(n_i)_i}(F)$ if and only if the characteristic polynomial admits a factorization $\chi(\gamma) = \prod_i \chi_i$ over F with $\deg \chi_i = n_i$.

4. Define a *lattice in* F^n to be a free, finitely-generated $\mathcal{O}$-submodule of F^n of rank n.

   (a) (Less important, feel free to use references or skip). Show that any free, finitely generated $\mathcal{O}$-submodule of F^n has rank $\leq n$. So lattices are the *maximal* free finitely-generated $\mathcal{O}$-submodules in F^n.

   (b) Construct a natural bijection between the set of such lattices and the quotient $\mathrm{GL}_n(F)/\mathrm{GL}_n(\mathcal{O})$. This object is called the *affine Grassmannian*, because it serves as the affine analog of a Grassmannian.

   (c) Can you give a similar interpretation of the quotient $\mathrm{GL}_n(F)/I$? (Hint: this is called the *affine flag variety*. A lattice is fundamentally an infinite-dimensional object, but you shouldn't need to specify an infinite amount of them.)

   (d) What is the analog of the above for the symplectic group or the orthogonal group? That is, can you give geometric interpretations of $G(F)/G(\mathcal{O})$ in these cases?

5. Fix the standard lattice $L_0 = \mathcal{O}^n \subset F^n$. Define the *type* of a lattice M to be the sequence of weakly increasing integers $(k_1, \ldots, k_n)$, $k_i \leq k_{i+1}$, defined such that $M \subset \pi^{k_1} L_0$ and

$$(\pi^{k_1} L_0)/M \cong \mathcal{O}/\pi^{k_1-k_1}\mathcal{O} \oplus \mathcal{O}/\pi^{k_2-k_1}\mathcal{O} \oplus \cdots \oplus \mathcal{O}/\pi^{k_n-k_1}\mathcal{O}.$$

   (a) Using the structure theory of modules over a PID, show that such an isomorphism necessarily exists and that the type is well-defined.

   (b) Every lattice M can be written as gL_0 for an element $g \in \mathrm{GL}_n(F)$. Show that (I think this is true) the lattice M has type $(k_1, \ldots, k_n)$ if and only if g lies in the same $\mathrm{GL}_n(\mathcal{O})\backslash \mathrm{GL}_n(F)/\mathrm{GL}_n(\mathcal{O})$-orbit as

$$\begin{pmatrix} x^{k_1} & & \\ & \ddots & \\ & & x^{k_n} \end{pmatrix}.$$

   (c) Did we ever figure out how to classify orbits for $\mathrm{Sp}_{2n}(\mathcal{O})\backslash \mathrm{Sp}_{2n}(F)/\mathrm{Sp}_{2n}(\mathcal{O})$? Can we get some down-to-earth definition of "type" that matches these orbits?

6. Let γ be diagonal with distinct entries. Prove that for suitable normalizations of Haar measures, that the orbital integral $O_\gamma(1_{\mathfrak{g}(\mathcal{O})})$ can be interpreted as counting a certain set of lattices up to an action of $\mathbb{Z}^n \cong \{t^{\vec{\lambda}} : \vec{\lambda} \in \mathbb{Z}^n\}$ (Hint: use the decomposition from problem 2. Translate the condition $g^{-1}\gamma g \in \mathfrak{g}(\mathcal{O})$ into a condition about the lattice $L = g\mathcal{O}^n$)

7. Do some reading about finite extensions of local fields (many good references exist. Chapter 2 of Neukirch's "Algebraic Number Theory" is one such). Prove that a local field has a unique unramified extension of each degree. Find *all* quadratic extensions of $\mathbb{F}_3((t))$ and $\mathbb{F}_5((t))$.

8. In Sp_{2n}, the standard parabolic subgroups are upper triangular groups defined by *symmetric* partitions of n. For example, the standard parabolics in Sp_4 (in addition to B and G itself) are

$$P_1 = \begin{pmatrix} \star & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & 0 & 0 & \star \end{pmatrix} \qquad P_2 = \begin{pmatrix} \star & \star & \star & \star \\ \star & \star & \star & \star \\ 0 & 0 & \star & \star \\ 0 & 0 & \star & \star \end{pmatrix}.$$

The Levi subgroups are the corresponding block-diagonal groups. Can you describe the isomorphism types of these Levis (e.g. write them as a product of smaller matrix groups)? Find the Levi decompositions $P_i = M_i \ltimes N_i$ for these specific subgroups. Can you describe the Haar measures on the two standard parabolic subgroups?

9. Let L_1, L_2 be any two lattices in F^n.

   (a) Prove that for any two lattices, there exists a lattice L contained in each of them which has finite codimension in each of them (in the sense that L_i/L is a finite-dimensional k-vector space).

   (b) Show that if $L_1 \subset L_2$, then

   $$[L_1 : L_2] = \dim(L_1/L_2)$$

   is finite.

   (c) Using the first part of this problem, define a well-defined and finite notion of "index" $[L_1 : L_2]$ for *any two lattices*, not necessarily contained in one another.

   (d) Define the *determinant* of a lattice L as follows. Represent the lattice as $g\mathcal{O}^n$ for some $g \in G(F)$, and define

   $$\det(L) = v(\det(g)) \in \mathbb{Z}.$$

   Show this is well-defined.[1]

   What can you say about the relationship between the determinant of lattices and the index you defined above?

10. First for $G = \mathrm{GL}_n$, and then for any other groups you'd like to try, prove the *Birkhoff decomposition* (no idea if this name is standard):

$$G(F) = \bigsqcup_{\lambda_1 \leq \cdots \leq \lambda_n} G(k[t^{-1}])t^{\lambda}G(\mathcal{O}).$$

Note that the left side consists of *polynomials* in t^{-1}, not power series.

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[1] By the way, there is a fancier way to think about this using exterior powers of modules: by functoriality of the n-th exterior power, there is a free $\mathcal{O}$-submodule $\bigwedge^n L \subset \bigwedge^n F^n \cong F$. But every (nonzero) free $\mathcal{O}$-submodule of F is equal to $\mathcal{O}t^k$ for some $k \in \mathbb{Z}$, so we get a well-defined integer.