QUALIFYING EXAM SYLLABUS

CHRISTOPHER EUR

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Committee: David Eisenbud, Bernd Sturmfels, Fraydoun Rezakhanlou (Chair), Ruzena Bajcsy

1. **Major Topic: Algebraic Geometry (Algebra)**

1.1. **Schemes.** affine, projective, reduced, irreducible, integral, regular, Noetherian, fiber product, varieties, blow-ups

1.2. **Morphisms.** open and closed embeddings, affine, finite, finite-type, separated, proper, valuative criteria, projective, flat, rational, dominant

1.3. **Sheaves.** coherent, locally free, invertible, ample, very-ample, twisting, maps to \( \mathbb{P}^n \), differentials

1.4. **Divisors.** Weil divisors, Cartier divisors, divisor class group and Picard group

1.5. **Cohomology.** sheaf cohomology, Čech cohomology, cohomology of projective space, Hilbert polynomials, Grothendieck vanishing theorem, Serre’s criterion for affineness, Serre duality (statement)

1.6. **Curves.** Bezout’s theorem, Riemann-Roch, Hurwitz formula, canonical embeddings, elliptic curves, hyperelliptic curves, Clifford’s theorem

**Reference:** Hartshorne, *Algebraic Geometry*. Chapters I.1–7, II.1–8, III.2–5,7,9, IV.1–5

2. **Major Topic: Commutative Algebra (Algebra)**

2.1. **Rings and modules.** localization, associated primes, primary decomposition, integral extension, going-up and going-down theorems, Noether normalization, Nakayama’s lemma, Nullstellensatz, Artin-Rees lemma, Krull intersection theorem, flatness

2.2. **Dimension theory.** Krull dimension, Hilbert-Samuel polynomials, Artin rings and finiteness results, fundamental theorem of dimension theory, Krull’s principal ideal theorem, regular local ring, Gröbner bases

2.3. **Homological methods.** Projective and injective resolutions, Ext and Tor, Koszul complexes, regular sequences, depth, Cohen-Macaulay rings, Hilbert syzygy theorem, projective dimension, Auslander-Buchsbaum formula

**Reference:** Eisenbud, *Commutative Algebra*. Chapters 1–6, 9–13, 15–19
3. MINOR TOPIC: COMPLEX ANALYSIS (ANALYSIS)


Reference: Stein & Shakarchi, Complex Analysis. Chapters 1-3, 8.
4. Transcript

Here is a list of questions I remember from my qual exam. E,S,R,B for Eisenbud, Sturmfels, Rezakhanlou, and Bajcsy.

1. (E) Can you tell us about the relationship between line bundles and divisors?
2. (E) How are line bundles related to maps to projective space?
3. (E) Suppose we have a map \( \varphi : X \to \mathbb{P}^n \). Compose this with a projection \( \mathbb{P}^n \to \mathbb{P}^m \) to obtain a map \( \psi : X \to \mathbb{P}^m \) (possibly not defined on all of \( X \)). What can you say about how the line bundles giving maps \( \varphi \) and \( \psi \) are related?
4. (E) Write down an example of what we have just discussed. Why don’t you just do the twisted cubic.
5. (S) What can you say about complete intersection of a cubic and a quadric in \( \mathbb{P}^3 \)?
6. (S) Project this curve away from a general point in \( \mathbb{P}^3 \) onto a \( \mathbb{P}^2 \). Many many singularities do you expect the image curve to have?
7. (E) Well, curves can all be embedded in \( \mathbb{P}^3 \). Can you generalize this to \( d \)-dimensional varieties, and tell us why?
8. (S) Suppose I instead have a surface in \( \mathbb{P}^3 \) and project onto a \( \mathbb{P}^2 \). Can you say things about its ramification locus?
9. (E) Why don’t we actually do that for curves first. Tell us the story of ramification points in the curves case.
10. (E) Can you prove the Hurwitz’s formula from the exact sequence of sheaf of differentials you wrote down? Why don’t you try computing each of their Euler characteristics?
11. (E) What does Riemann-Roch say? Can you guess Riemann’s precursor to the formula? Here is a hint: It was an inequality and Roch, who was Riemann’s student, finished the formula
12. (E) How would Riemann have phrased Riemann-Roch in his time?
13. (S) Let \( I = (xz, xw + yz, yw) \) be an ideal in \( S = k[x, y, z, w] \). Can you compute its Grobner basis?
14. (S) Is the ideal reduced? If not, what is its radical? Can you write down the primary decomposition of the radical?
15. Is the radical Cohen-Macaulay?
16. (S) What are the associated primes of \( I \)? Is primary decomposition unique? Can you write down a primary decomposition of \( I \)?
17. (S) What does Auslander-Buchsbaum theorem say? Why does it apply here?
18. (S) What is the depth of \( S/I \) of \( S/\sqrt{I} \)?
19. (R) Can you prove the Cauchy integral formula?
20. (R) Can you prove it without using Goursat’s theorem. That is, just use Green’s theorem.
21. (R) What does open mapping theorem say? Can you sketch a proof of it?
22. (R) What does Riemann mapping theorem say? Can you sketch a proof?
23. (R) What are all the automorphisms of a simply-connected open proper subset of \( \mathbb{C} \)? Why? What is its real dimension?