EXAMPLES FOR "DIVISORS ON MATROIDS AND THEIR VOLUMES"

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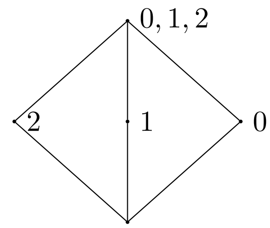
All the matroids in the examples are realizable to illustrate the algebro-geometric connections. Let's first start with a sanity check:

Example 1. Let $M := U_{2,3}$, the uniform matroid of rank 2 on 3 elements. The lattice of flats $\mathscr{L}_M$ is



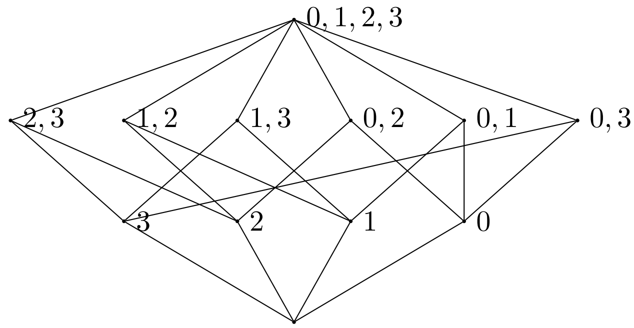
and the Chow ring $A(M)$ is $A(M) = \dfrac{\mathbb{Z}[x_0, x_1, x_2]}{(x_2x_1, x_2x_0, x_1x_0, x_1 - x_0, x_2 - x_0)}$, which is isomorphic to $\mathbb{Z}[x_0]/(x_0^2)$. As the complement of the hyperplane arrangement $\mathcal{A}_M$ is $\mathbb{P}^1 \setminus \{p, q, r\}$ for p, q, r distinct points, the wonderful compactification is $Y_M = \mathbb{P}^1$, whose Chow ring is also $\mathbb{Z}[h]/(h^2)$ where h is the class of a point on $\mathbb{P}^1$. The volume polynomial is

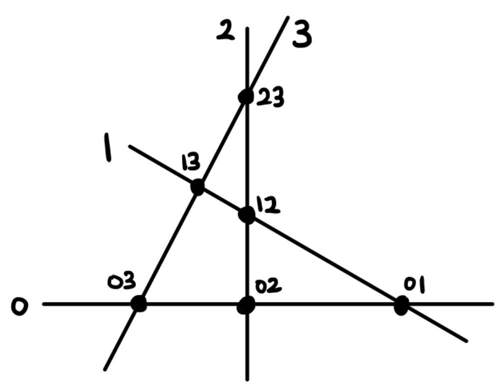
$$VP_M = t_0 + t_1 + t_2.$$

As expected, a divisor $a_0x_0 + a_1x_1 + a_2x_2 = (a_0 + a_1 + a_2)h$ has volume $a_0 + a_1 + a_2$ (when the sum is ≥ 0). The distinguished divisor $x_0 + x_1 + x_2$ has volume $3 = 3^{2-1}$.

The next two examples are of rank 3, whose wonderful compactifications are obtained from blowing-up points on $\mathbb{P}^2$. For these surfaces, we can easily compute the intersection numbers classically; see [Har77, §V.3] for reference on monoidal transformations.

Example 2. Let $M := U_{3,4}$. Its lattice of flats $\mathscr{L}_M$ and the hyperplane arrangement $\mathcal{A}_M \subset \mathbb{P}^2$ are





The wonderful compactification Y_M is given by blowing-up the six points $H_{0,1}, \dots, H_{2,3}$. The volume polynomial is

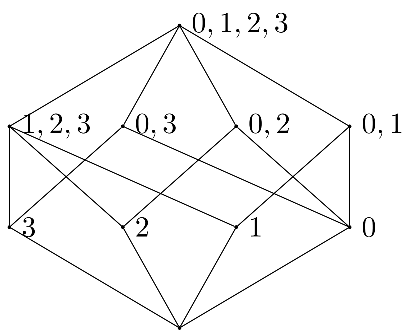
$$\begin{aligned} VP_M(\underline{t}) = \; & -2t_1^2 - 2t_3^2 - 2t_0^2 - 2t_2^2 + 2t_3t_{2,3} + 2t_2t_{2,3} - t_{2,3}^2 + 2t_1t_{1,3} + 2t_3t_{1,3} - t_{1,3}^2 + 2t_3t_{0,3} \\ & +2t_0t_{0,3} - t_{0,3}^2 + 2t_1t_{1,2} + 2t_2t_{1,2} - t_{1,2}^2 + 2t_0t_{0,2} + 2t_2t_{0,2} - t_{0,2}^2 + 2t_1t_{0,1} + 2t_0t_{0,1} - t_{0,1}^2. \end{aligned}$$

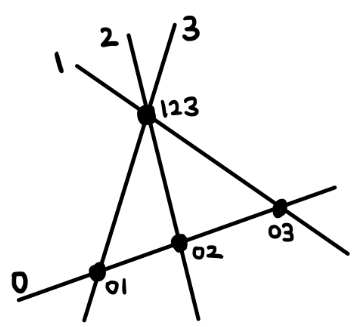
Notice that the coefficient of t_i^2 is -2 as expected from Corollary **??** since $M/\{i\} \simeq U_{2,3}$ so that $|\mu^1(M/\{i\})| = 2$, and the coefficients of $t_{i,j}^2$ is -1 in agreement with Proposition **??**. The rest of the coefficients are maximal chains, so the coefficient is $\binom{2}{1,1} = 2$. The volume of M is $(4)(-2)(1) + (12)(2)(2) + (6)(-4) = 16 = 4^2$, as expected from Theorem **??**.

Let $\pi : Y_M \to \mathbb{P}^2$ be the blow-down map, $\widetilde{H} := \pi^* H$ the pullback of the hyperplane class $H \subset \mathbb{P}^2$, and E_{ij}'s the exceptional divisors from the blown-up points. Then $\operatorname{Pic} Y_M = \mathbb{Z}\{\widetilde{H}, E_1, \ldots, E_6\}$, where intersection pairing of divisors are $E_i \cdot E_j = 0 \; \forall i \neq j$, $E_i \cdot \widetilde{H} = 0$, $\widetilde{H} \cdot \widetilde{H} = 1$, and $E_i \cdot E_i = -1$. Hence, $x_0 = \pi^* H_0 = \widetilde{H} + E_{01} + E_{02} + E_{03}$, so that $x_0^2 = 1 - 1 - 1 - 1 = -2$, as expected. Similarly, one computes that the volume of M is $4^2 = 16$. Alternatively, note that the map $Y_M \to \mathbb{P}(H^0(4\widetilde{H}))$ given by $4\widetilde{H}$ factors birational through $\mathbb{P}^2$ as the 4-tuple Veronese embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^{14}$, whose degree is 16.

A side remark: the complete linear system $|3\widetilde{H} - E_{01} - \cdots - E_{23}|$ defines a birational map $Y_M \to \mathbb{P}^3$ whose image is the Cayley nodal cubic surface (as $(3\widetilde{H} - E_{01} - \cdots - E_{23})^2 = 3$) . Indeed, as $3\widetilde{H} - E_{01} - \cdots - E_{23} = 3(x_0 + x_{0,1} + x_{0,2} + x_{0,3}) - (x_{0,1} + \cdots + x_{2,3}) = 3x_0 + 2x_{0,1} + 2x_{0,2} + 2x_{0,3} - x_{1,2} - x_{1,3} - x_{2,3}$, evaluating VP_M respectively gives $-2 \cdot 3^2 - 3 \cdot 2^2 - 3 \cdot 1^1 + 3 \cdot 2 \cdot 3 \cdot 2 = 3$.

Example 3. Let $M = U_{1,1} \oplus U_{2,3}$, another matroid of rank 3 on 4 elements, but not uniform as the above example. Its lattice of flats $\mathscr{L}_M$ and hyperplane arrangement $\mathcal{A}_M$ are





Its wonderful compactification $\pi : Y_M \to \mathbb{P}^2$ is the plane $\mathbb{P}^2$ blown-up at four points $H_{01}, H_{02}, H_{03}, H_{123}$. The volume polynomial is

$$\begin{aligned} VP_M(\underline{t}) = \; & -t_3^2 - t_2^2 - t_1^2 - 2t_0^2 + 2t_3 t_{0,3} + 2t_0 t_{0,3} - t_{0,3}^2 + 2t_2 t_{0,2} + 2t_0 t_{0,2} \\ & -t_{0,2}^2 + 2t_1 t_{0,1} + 2t_0 t_{0,1} - t_{0,1}^2 + 2t_3 t_{1,2,3} + 2t_2 t_{1,2,3} + 2t_1 t_{1,2,3} - t_{1,2,3}^2. \end{aligned}$$

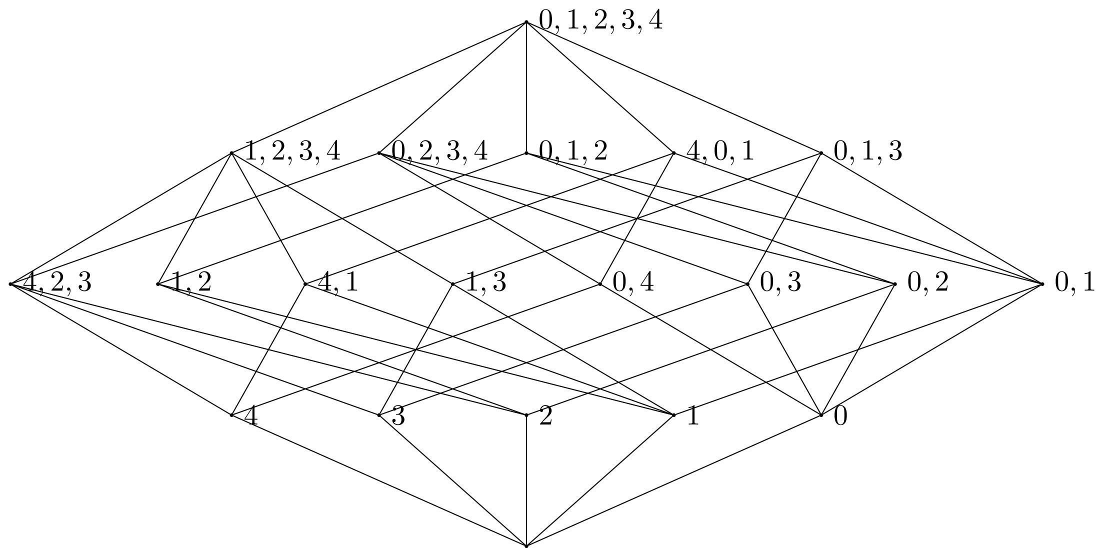
Notice that the coefficient of t_0^2 is -2 while those of t_i^2 $(i \neq 0)$ are -1, since $\mathscr{L}_{M/0} \simeq \mathscr{L}_{U_{2,3}}$ whereas $\mathscr{L}_{M/1} \simeq \mathscr{L}_{U_{2,2}}$ (the reduced chromatic polynomials of $U_{2,3}$ and $U_{2,2}$ are $t-2$ and $t-1$). The volume of M is $-5 + (4)(-2^2) + 9(2)(1 \cdot 2) = 15 < 16$.

Again, let $\widetilde{H} := \pi^* H$ the pullback of the hyperplane class H, and E_{ij} the exceptional divisors of from the blown-up points. As in the proof of Theorem **??**, the distinguished divisor D_M is $4\widetilde{H} - E_{123}$, whose volume is $(4\widetilde{H} - E_{123})^2 = 16 - 1 = 15$. Alternatively, the map $Y_M \to \mathbb{P}(H^0(4\widetilde{H} - E_{123}))$ factors birational through $\mathbb{P}^2$ as a rational map $\mathbb{P}^2 \to \mathbb{P}^{13}$ given by a graded linear system $L \subset H^0(\mathscr{O}_{\mathbb{P}^2}(4))$ consisting of quartics through the point H_{123}. Its image is the blow-up of a point in $\mathbb{P}^2$ embedded in $\mathbb{P}^{13}$ with degree 15. In summary,

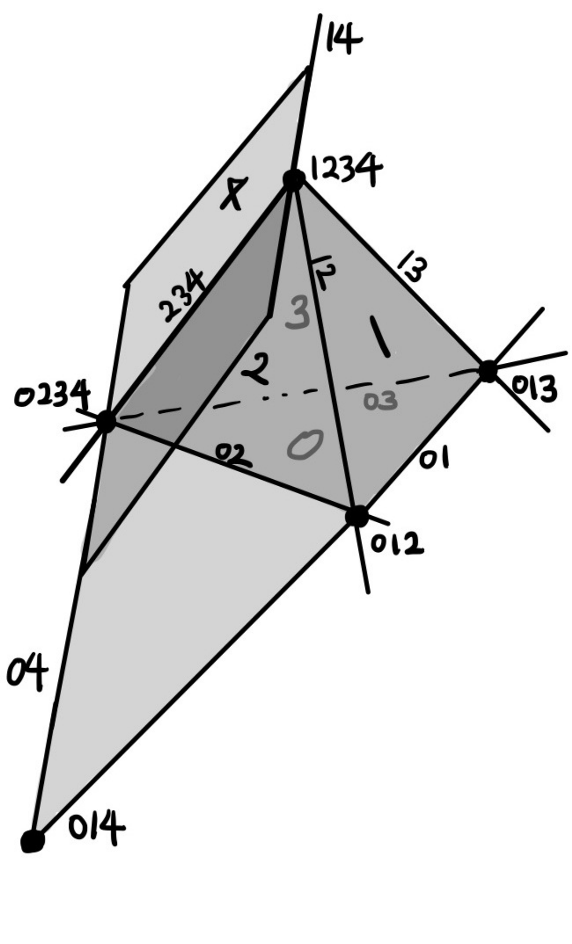
$$\begin{array}{ccccc} Y_M & \longrightarrow & \operatorname{Bl}_{H_{123}} \mathbb{P}^2 & \hookrightarrow & \\ & \searrow^{\pi} & \downarrow & & \searrow \\ & & \mathbb{P}^2 & \xhookrightarrow{Veronese} \mathbb{P}^{\binom{4+2}{2}-1} \dashrightarrow & \mathbb{P}^{13}. \end{array}$$

We feature one example of rank 4. Here, we can see that the computation of the volume polynomial and the intersection numbers already become somewhat nontrivial.

Example 4. Let $M := U_{2,2} \oplus U_{2,3}$. Its lattice of flats $\mathscr{L}_M$ is as follows.



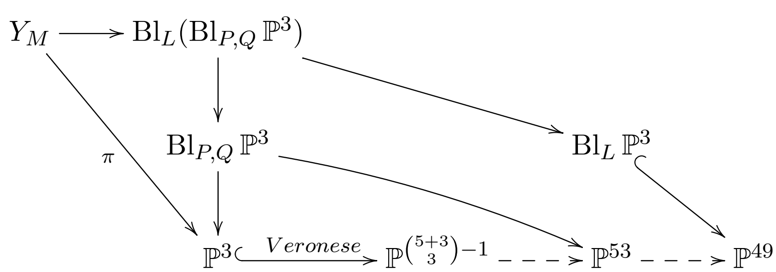
And its hyperplane arrangement $\mathcal{A}_M$ can be illustrated as



The wonderful compactification $\pi : Y_M \to \mathbb{P}^3$ is obtained by blowing-up the four points, then the strict transforms of the 8 lines. The volume polynomial is

$$\begin{aligned}
VP_M(\underline{t}) = \; & t_4^3 + t_3^3 + t_2^3 + 2t_1^3 + 2t_0^3 - 3t_4t_{4,1}^2 - 3t_1t_{4,1}^2 + 2t_{4,1}^3 - 3t_3t_{1,3}^2 - 3t_1t_{1,3}^2 + 2t_{1,3}^3 - 3t_2t_{1,2}^2 \\
& -3t_1t_{1,2}^2 + 2t_{1,2}^3 - 3t_4t_{0,4}^2 - 3t_0t_{0,4}^2 + 2t_{0,4}^3 - 3t_3t_{0,3}^2 - 3t_0t_{0,3}^2 + 2t_{0,3}^3 - 3t_2t_{0,2}^2 - 3t_0t_{0,2}^2 \\
& +2t_{0,2}^3 - 6t_1t_{0,1}^2 - 6t_0t_{0,1}^2 + 4t_{0,1}^3 - 3t_4t_{4,2,3}^2 - 3t_3t_{4,2,3}^2 - 3t_2t_{4,2,3}^2 + 2t_{4,2,3}^3 - 3t_4^2t_{4,0,1} \\
& -3t_1^2t_{4,0,1} - 3t_0^2t_{4,0,1} + 6t_4t_{4,1}t_{4,0,1} + 6t_1t_{4,1}t_{4,0,1} - 3t_{4,1}^2t_{4,0,1} + 6t_4t_{0,4}t_{4,0,1} + 6t_0t_{0,4}t_{4,0,1} \\
& -3t_{0,4}^2t_{4,0,1} + 6t_1t_{0,1}t_{4,0,1} + 6t_0t_{0,1}t_{4,0,1} - 3t_{0,1}^2t_{4,0,1} - 3t_4t_{4,0,1}^2 - 3t_1t_{4,0,1}^2 - 3t_0t_{4,0,1}^2 \\
& +t_{4,0,1}^3 - 3t_3^2t_{0,1,3} - 3t_1^2t_{0,1,3} - 3t_0^2t_{0,1,3} + 6t_3t_{1,3}t_{0,1,3} + 6t_1t_{1,3}t_{0,1,3} - 3t_{1,3}^2t_{0,1,3} \\
& +6t_3t_{0,3}t_{0,1,3} + 6t_0t_{0,3}t_{0,1,3} - 3t_{0,3}^2t_{0,1,3} + 6t_1t_{0,1}t_{0,1,3} + 6t_0t_{0,1}t_{0,1,3} - 3t_{0,1}^2t_{0,1,3} - 3t_3t_{0,1,3}^2 \\
& -3t_1t_{0,1,3}^2 - 3t_0t_{0,1,3}^2 + t_{0,1,3}^3 - 3t_2^2t_{0,1,2} - 3t_1^2t_{0,1,2} - 3t_0^2t_{0,1,2} + 6t_2t_{1,2}t_{0,1,2} + 6t_1t_{1,2}t_{0,1,2} \\
& -3t_{1,2}^2t_{0,1,2} + 6t_2t_{0,2}t_{0,1,2} + 6t_0t_{0,2}t_{0,1,2} - 3t_{0,2}^2t_{0,1,2} + 6t_1t_{0,1}t_{0,1,2} + 6t_0t_{0,1}t_{0,1,2} \\
& -3t_{0,1}^2t_{0,1,2} - 3t_2t_{0,1,2}^2 - 3t_1t_{0,1,2}^2 - 3t_0t_{0,1,2}^2 + t_{0,1,2}^3 - 3t_4^2t_{1,2,3,4} - 3t_3^2t_{1,2,3,4} - 3t_2^2t_{1,2,3,4} \\
& -6t_1^2t_{1,2,3,4} + 6t_4t_{4,1}t_{1,2,3,4} + 6t_1t_{4,1}t_{1,2,3,4} - 3t_{4,1}^2t_{1,2,3,4} + 6t_3t_{1,3}t_{1,2,3,4} + 6t_1t_{1,3}t_{1,2,3,4} \\
& -3t_{1,3}^2t_{1,2,3,4} + 6t_2t_{1,2}t_{1,2,3,4} + 6t_1t_{1,2}t_{1,2,3,4} - 3t_{1,2}^2t_{1,2,3,4} + 6t_4t_{4,2,3}t_{1,2,3,4} + 6t_3t_{4,2,3}t_{1,2,3,4} \\
& +6t_2t_{4,2,3}t_{1,2,3,4} - 3t_{4,2,3}^2t_{1,2,3,4} - 3t_4t_{1,2,3,4}^2 - 3t_3t_{1,2,3,4}^2 - 3t_2t_{1,2,3,4}^2 - 3t_1t_{1,2,3,4}^2 + t_{1,2,3,4}^3 \\
& -3t_4^2t_{0,2,3,4} - 3t_3^2t_{0,2,3,4} - 3t_2^2t_{0,2,3,4} - 6t_0^2t_{0,2,3,4} + 6t_4t_{0,4}t_{0,2,3,4} + 6t_0t_{0,4}t_{0,2,3,4} \\
& -3t_{0,4}^2t_{0,2,3,4} + 6t_3t_{0,3}t_{0,2,3,4} + 6t_0t_{0,3}t_{0,2,3,4} - 3t_{0,3}^2t_{0,2,3,4} + 6t_2t_{0,2}t_{0,2,3,4} + 6t_0t_{0,2}t_{0,2,3,4} \\
& -3t_{0,2}^2t_{0,2,3,4} + 6t_4t_{4,2,3}t_{0,2,3,4} + 6t_3t_{4,2,3}t_{0,2,3,4} + 6t_2t_{4,2,3}t_{0,2,3,4} - 3t_{4,2,3}^2t_{0,2,3,4} - 3t_4t_{0,2,3,4}^2 \\
& -3t_3t_{0,2,3,4}^2 - 3t_2t_{0,2,3,4}^2 - 3t_0t_{0,2,3,4}^2 + t_{0,2,3,4}^3.
\end{aligned}$$

Its volume is 112, which is the smallest for simple matroids of rank 4 on 5 elements. Let $\widetilde{H} = \pi^*H$ be the pullback of hyperplane $H \subset \mathbb{P}^3$ again, and let E_F's be the exceptional divisors from the blow-ups. The distinguished divisor D_M, again following the proof of Theorem **??** is $5\widetilde{H} - E_{0234} - E_{1234} - E_{234}$. Let's call P, Q the two points E_{0234}, E_{1234}, and L the line E_{234}. Then the map given by D_M on Y_M factors through a rational map on $\mathbb{P}^3$ as follows. First, consider the rational map given by $L \subset H^0(\mathscr{O}_{\mathbb{P}^3}(5))$ consisting of divisors through the two points P, Q. Then, consider the map from $\operatorname{Bl}_{P,Q}\mathbb{P}^3$ given by divisors in $H^0(\operatorname{Bl}_{P,Q}\mathbb{P}^3)$ passing containing the strict transform of the line L. The image of the map is the blow-up of L in $\mathbb{P}^3$ embedded in $\mathbb{P}^{49}$ with degree 112. In summary, we have



References

[Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.