This document summarizes results in Bennett’s *Affine and Projective Geometry* by more or less following and rephrasing “Faculty Senate Affine Geometry” by Paul Bamberg in a more mathematically conventional language (so it does not use terms “senate, faculty, committee, etc.”) Figures are all taken from Bennett’s book.

1 Brief Recap of Euclidean Geometry

**Definition 1.1** (Euclid’s Five Axioms of Geometry). Euclid stated five axioms for Euclidean geometry of the plane:

1. A straight line can be drawn between any two points.
2. A line can be extended indefinitely in either direction.
3. A circle can be described with any point as center and any segment as radius.
4. All right angles are equal.
5. Through a point not on a line there exists a unique line parallel to the given line.

**Observation 1.2.** A set of axioms describing some mathematical structure would be useless if there isn’t a concrete example which satisfies the axioms. In our this case, the space $\mathbb{R}^2$ satisfies all five axioms as above. The points are $(x, y) \in \mathbb{R}^2$ and a line is $L = \{(x, y) \in \mathbb{R}^2 \mid ax + by = c\}$ for some $a, b, c \in \mathbb{R}$, and so forth.

**Remark 1.3.** We can immediately start asking what kind of “geometry” $F^2$ for any field $F$ should have. This is one of the motivations behind trying to axiomatize what an “affine plane” should be.

The following two are classical theorems in Euclidean plane geometry that will turn out to be significant for our purposes later:

**Theorem 1.4** (Desargues). *Desargues theorem comes in two parts, which can really be said in one statement. Let’s denote by $L(A, B)$ the line through points $A, B$.*

- Suppose that either $L(A, A') \parallel L(B, B') \parallel L(C, C')$ or $L(A, A') \cap L(B, B') \cap L(C, C') = P$. If $L(A, B) \parallel L(A', B')$, and $L(A, C) \parallel L(A', C')$, then $L(B, C) \parallel L(B', C')$. 

Theorem 1.5 (Pappus). Let $l, m \in \mathcal{L}$ such that $l \cap m = O$, and let $P, Q, R \in l$, $S, T, U \in m$ such that $l(P, T) \parallel l(Q, U)$ and $l(Q, S) \parallel l(R, T)$. Then $l(P, S) \parallel l(R, U)$.

2 The Affine Plane

What we defined as the affine plane in class is really a Desarguesian and Pappus-ian affine plane. Here we define what just an affine plane is, and show that addition of points and be defined,

Definition 2.1. An affine plane consists of the data $(\mathcal{P}, \mathcal{L})$—the set of “points” $\mathcal{P}$ and the set of “lines” $\mathcal{L} \subset \mathcal{P}(\mathcal{P})$—satisfying the following axioms:

1. $\forall A, B \in \mathcal{P}$, $\exists! l \in \mathcal{L}$ s.t. $A, B \in l$. We denote this line as $l(a, b)$.

2. Given $l \in \mathcal{L}$ and $A \notin l$, $\exists! l' \in \mathcal{L}$ such that $A \in l'$ and $l' \cap l = \emptyset$. We say that $l, l'$ are parallel when $l \cap l' = \emptyset$ or $l = l'$, and we denote it as $l \parallel l'$.

3. $|l| \geq 2$ $\forall l \in \mathcal{L}$ and $|\mathcal{L}| \geq 2$. (This eliminates uninteresting cases like: affine line and single point with many lines)

With these axioms, we can already deduce some nontrivial theorems:
Lemma 2.2. Let $(\mathcal{P}, \mathcal{L})$ be an affine plane.

1. For any two distinct $l, l' \in \mathcal{L}$, either $l \cap l' = \emptyset$ or $l \cap l' = \{P\}$ (a single point).

2. Suppose $m$ intersects $l$ and $l \parallel l'$ ($m \neq l$), then $m$ intersects $l'$.

3. $\parallel$ is an equivalence relation on $\mathcal{L}$.

proof) Given in class for 1. and 2., and 3. follows from 2.

Theorem 2.3. Let $(\mathcal{P}, \mathcal{L})$ be an affine plane. Suppose $l, r \in \mathcal{L}$. There exists a bijective map $f : l \rightarrow r$. That is, there exists a bijective correspondence between points of $l$ and points of $r$.

Definition 2.4. Now suppose that for some $l \in \mathcal{L}$, $|l| < \infty$. Then by above theorem we have that for all $k \in \mathcal{L}$, $|k| = n$ are finite and are all same. In this case, we say $(\mathcal{P}, \mathcal{L})$ is an affine plane of order $n$.

Corollary 2.5. Let $(\mathcal{P}, \mathcal{L})$ be an affine plane of order $n$. Then from Theorem 2.3 we immediately have the following:

1. $|l| = n \forall l \in \mathcal{L}$.

2. Fix $A \in \mathcal{P}$. The number of $l \in \mathcal{L}$ such that $A \in l$ is $n + 1$.

3. A pencil is an equivalence class of $\parallel$. There are $n + 1$ pencils each containing $n$ lines, and thus $|\mathcal{L}| = n(n + 1)$.

4. $|\mathcal{P}| = n^2$

Remark 2.6. Note that everything so far is direct result of the three axioms we assumed for an affine plane. We did not need notions regarding Desargues or Pappus yet.
3 The Desarguesian + Pappusian Affine Plane

We now wish to define how to add two points on a line after fixing an origin and how to multiply two points on a line after fixing an origin and the unit point. For these operations to behave well (well-defined, commutative, etc.) we need additional structure on the affine plane so far, and those additional axioms are exactly those saying that the Desargues and Pappus theorems hold. That is, from here on, we assume that by an “affine plane” we mean one in which Desargues and Pappus theorems are satisfied.

**Definition 3.1** (addition). Let $A, C$ be points on a line $l$. Fix a point $O \in l$. We define adding $A$ to $C$ with respect to $O$, denoted $A + C$, by the following construction:

1. Take any point $B \notin l$ and let $m \ni B$ be the unique line parallel to $l$.

2. Let $k$ be a line through $A$ parallel to the line through $O, B$, and let $k \cap m = D$.

3. Now let $n$ be a line through $D$ parallel to the line through $B, C$.

4. Define $A + C$ as the intersection of $n$ and $l$.

**Proposition 3.2.** The above definition is well-defined. That is, $A + C$ (w.r.t. $O$) is independent of the choice of “$B$.”

Proof) Draw the diagram and be convinced using Desargues (may need to use it twice)

**Proposition 3.3.** Addition is associative. That is, given $A, C, E \in l$ and a fixed $O \in l$, $(A+C) + E = A + (C + E)$.
Proposition 3.4. Let $l \in \mathcal{L}$, and fix $O \in l$ so that $+$ is defined on $l$. Then, $\forall A \in l \ A + O = A$ and $\forall A \in l \ \exists V \in l$ s.t. $A + V = O$. That is, $(l, +)$ is a group.

Proof) Again, draw the diagram and be convinced.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{diagram1.png}
\end{figure}

Theorem 3.5 (Converse of Desargues). If $L(A, B) \parallel L(A', B')$, $L(A, C) \parallel L(A', C')$, and $L(B, C) \parallel L(B', C')$, then either $L(A, A') \parallel L(B, B') \parallel L(C, C')$ or $L(A, A') \cap L(B, B') \cap L(C, C') = P$.

Proof) Suffices to show that if two lines meet then the third one meets at the same point too.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{diagram2.png}
\end{figure}

Corollary 3.6. Addition is commutative.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{diagram3.png}
\end{figure}
Definition 3.7 (Multiplication). Let $l \in \mathcal{L}$ and fix $O \in l$ and $I \in l$ ($O \neq I$). Given $A, C \in l$, we define the product of $A, C$ by the following procedure:

1. Choose a $B \notin l$. Draw, $l(O, B)$, $l(I, B)$, and draw a line parallel to $l(I, B)$ through $A$, which intersects $l(O, B)$ at $D$.
2. Draw $l(B, C)$, and let $k$ be the parallel line through $D$.
3. $k \cap l$ is defined as $A \cdot C$.

Proposition 3.8. Multiplication as defined above is well-defined.

Proof) Again draw the diagram, and see that we just use the Desargues twice.

Proposition 3.9. Multiplication is associative. Moreover, $A \cdot I = A$ and for $A \neq O$, there exists $B$ such that $AB = I$. Hence, $l - \{O\}$ with $O, I$ fixed and with multiplication is a group.

Proof) Let’s trust Paul and believe that multiplication is associative. For the right identity and right inverse, just draw the diagrams.

Theorem 3.10. Multiplication is commutative. Moreover, distributivity holds for the addition and multiplication we have defined. Hence, $l$ becomes a field.

Proof) For the sake of sanity, we only did commutativity in class, which is quite easy after drawing the diagram and citing A5.

4 Coordinatization

Now, we have enough tools to put a coordinate system on a given affine space $(\mathcal{P}, \mathcal{L})$.

Construction 4.1. Given an (Desarguesian & Pappusian) affine space $(\mathcal{P}, \mathcal{L})$, we coordinatize it as follows:

1. Fix an arbitrary point as $(0, 0) := O \in \mathcal{P}$, and take two distinct lines $x, y \in \mathcal{L}$ (i.e. the $x$-axis and $y$-axis) that contains $O$. 

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2. On each line, pick a point \((1, 0) \in x\) and \((0, 1) \in y\).

3. Now, for each point \(A \in \mathcal{P}\), by drawing the parallel lines to \(x\) and \(y\) going through \(A\) and looking at the intersections with \(x\) and \(y\), we have that \(\mathcal{P} \simeq x \times y\) as sets.

The construction above essentially proves the following:

**Theorem 4.2.** For any affine plane \((\mathcal{P}, \mathcal{L})\) and axes \(x = l(O, I)\) and \(y = l(O, I')\) in \(\mathcal{L}\), the points \(\mathcal{P}\) are in one-one correspondence with the ordered pairs in \(x \times y\).