Kähler-Einstein Metrics on Fano Manifolds

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1 Introduction

One way to obtain information about the geometry of a complex manifold is by looking at the metrics it admits. When do complex manifolds admit metrics which give them a special structure? For example, requiring a metric of constant bisectional curvature puts a strong restriction on the manifold; in the case of a complete Kähler manifold, it then has universal cover $\mathbb{CP}^n, \mathbb{C}^n$ or the unit ball in $\mathbb{C}^n$ and its metric pulls back to the canonical metrics on these covers, up to scaling, [Tia00, Theorem 1.12]. So one may look for a slightly weaker condition, such as the existence of Kähler-Einstein metrics. These are Kähler metrics for which the Ricci form is proportional to the Kähler form. Kähler-Einstein (KE) metrics provide a special solution to the Einstein equation, which describes how space-time curves as a result of gravitation from mass and energy. An Einstein metric solves the Einstein equation in a vacuum, [Bes87].

In order for KE metrics to exist on a compact complex manifold, the first Chern class must be definite. Under this condition, the first Chern class separates complex manifolds into three cases. M always admits a KE metric when $c_1(M) \leq 0$. In the case of positive first Chern class, M is called a Fano manifold and does not always admit a Kähler-Einstein metric. Obstructions include the vanishing of the Calabi-Futaki invariant, asymptotic Chow stability and K-stability. The answer to when KE metrics exist is known for Fano surfaces. These surfaces have been classified by N. Hitchin in [Hit75]. $\mathbb{CP}^1 \times \mathbb{CP}^1$ and $\mathbb{CP}^2$ admit Kähler-Einstein metrics. If $M = \mathbb{CP}^2(p_1, \ldots, p_k)$, the blow up of $\mathbb{CP}^2$ in k points in general position, then M does not admit a Kähler-Einstein metric for $k \in \{1, 2\}$ and it does for $k \in \{3, \ldots, 8\}$ ([Tia00, pg 87]).

The layout of this essay is as follows: in Section 1 I will give background for Kähler manifolds and Chern classes. In Section 2 I will state the Calabi-Yau theorem (without proof) and note some corollaries. In Section 3 I will introduce the Calabi-Futaki invariant, show $\mathbb{CP}^2(p)$ and $\mathbb{CP}^1 \times \mathbb{CP}^1(p)$ do not admit Kähler-Einstein metrics, and discuss the other Fano surfaces. In Section 4 I will introduce asymptotic Chow stability and describe an equivalence between Chow polystability and balanced varieties as given in Wang [Wan04], and state the theorem of Donaldson [Don01] that the existence of a constant scalar curvature Kähler (cscK) metric implies there is a sequence of balanced metrics converging to the cscK metric, when the automorphism group is discrete. In Section 5 I will define Tian's K-stability and describe his proof [Tia97] that Kähler-Einstein implies weakly K-stable.

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1.1 Kähler metrics

The background material of this section is similar to that in [Tia00].

Let (M, g) be a smooth Riemannian manifold. An almost complex structure $J : TM \to TM$ is an endomorphism of the tangent bundle such that $J^2 = -id$. The Nijenhuis tensor $N(J)$ is

$$N(J) : TM \times TM \to TM$$

$$N(J)(u, v) = [u, v] + J[Ju, v] + J[u, Jv] - [Ju, Jv], \quad u, v \in Vect(M)$$

M is a complex manifold if it admits an almost complex structure J which is also integrable, meaning J is induced from multiplication by i on the holomorphic tangent bundle $T^{1,0}M \subset T_{\mathbb{C}}M := TM \otimes \mathbb{C}$, defined below. By a theorem of Newlander and Nirenberg, J is integrable if and only if $N(J) = 0$.

Definition 1. J is compatible with g if

$$g(Ju, Jv) = g(u, v), \quad \forall u, v \in Vect(M)$$

Definition 2. The Kähler form of g is defined as

$$\omega_g(u, v) := -g(u, Jv)$$

Remark 3. The Kähler form is alternating by compatibility of J:

$$\begin{aligned}\omega_g(v, u) &= -g(v, Ju) = -g(Jv, -u) \\ &= g(u, Jv) = -\omega_g(u, v)\end{aligned}$$

Let ∇ denote the Levi-Civita connection on M. This is the unique torsion-free connection on M such that $\nabla g = 0$.

Definition 4. A Riemannian manifold (M, g) with a compatible almost complex structure J is a Kähler manifold if $\nabla J = 0$. Then g is a Kähler metric.

Throughout this essay, "Kähler metric ω_g" means "Kähler form ω_g corresponding to Kähler metric g".

Remark 5. Note that $\nabla J = 0 \implies N(J) = 0$. This can be seen as follows: ∇ is symmetric so $[u, v] = \nabla_u v - \nabla_v u$. Further, $J \in \Gamma(M, End(TM))$ means

$$(\nabla_X J)Y := \nabla_X(JY) - J(\nabla_X Y) \tag{1}$$

for all vector fields X, Y on M. Thus taking $X = Ju$, $Y = v$ and $X = v, Y = Ju$ respectively,

$$\begin{aligned}J\nabla_{Ju}v &= \nabla_{Ju}(Jv) - (\nabla_{Ju}J)v \\ J\nabla_v(Ju) &= -\nabla_v u - (\nabla_v J)Ju\end{aligned}$$

and so

$$\begin{aligned}N(J)(u, v) &= [u, v] + J[Ju, v] + J[u, Jv] - [Ju, Jv] \\ &= \nabla_u v - \nabla_v u + J(\nabla_{Ju}v - \nabla_v Ju) + J(\nabla_u Jv - \nabla_{Jv}u) - (\nabla_{Ju}Jv - \nabla_{Jv}Ju) \\ &= (\nabla_u v - \nabla_v u) - J(\nabla_v Ju - \nabla_u Jv) - (\nabla_{Ju}J)v + (\nabla_{Jv}J)u \\ &= (\nabla_v J)Ju - (\nabla_u J)Jv - (\nabla_{Ju}J)v + (\nabla_{Jv}J)u = 0\end{aligned}$$

if $\nabla J = 0$. So a Kähler manifold is a complex manifold. Equivalently, a Kähler manifold could be defined as a complex manifold with metric g such that the induced Kähler form ω_g is d-closed. Then g is a Kähler metric. See [Tia00, Prop 1.5] for this equivalence.

1.1.1 Holomorphic tangent bundle

Definition 6. Assume M is a Kähler manifold. J induces a splitting of $T_{\mathbb{C}}M$ into eigenspaces $T^{1,0}M \oplus T^{0,1}M$ corresponding to eigenvalues $+i, -i$ respectively, called the holomorphic and antiholomorphic tangent bundles.

We can extend g $\mathbb{C}$-linearly to $g_{\mathbb{C}}$ on $T_{\mathbb{C}}M$. Note that $TM \cong T^{1,0}M$ as real vector bundles; if M has local real coordinates $x_1, \ldots, x_{2n}$ and local complex coordinates $z_j := x_j + ix_{n+j}$, $1 \leq j \leq n$, then this isomorphism is given by

$$\frac{\partial}{\partial x_i} \mapsto \frac{\partial}{\partial z_i} := \frac{1}{2}\left(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial x_{n+i}}\right)$$
$$\frac{\partial}{\partial x_{n+i}} \mapsto i\frac{\partial}{\partial z_i}$$

for $1 \leq i \leq n$. $T^{0,1}M$ is generated by $\frac{\partial}{\partial \overline{z}_i} := \frac{1}{2}\left(\frac{\partial}{\partial x_i} + i\frac{\partial}{\partial x_{n+i}}\right)$ for $1 \leq i \leq n$.

Define a hermitian inner product on $T^{1,0}M$ by $h(u,v) = g(u,\overline{v})$. In local complex coordinates, $h = \sum_{i,j} g_{i\overline{j}} dz_i \otimes d\overline{z_j}$, where $g_{i\overline{j}} = g_{\mathbb{C}}\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z_j}}\right)$. Then from the definition of the Kähler form, ω_g is locally

$$\omega_g = \frac{i}{2}\sum_{i,j} g_{i\overline{j}} dz_i \wedge d\overline{z_j} \tag{2}$$

The metric h is hermitian since g is symmetric and $\overline{\frac{\partial}{\partial z_i}} = \frac{\partial}{\partial \overline{z}_i}$, therefore

$$\overline{h_{ij}} = \overline{g_{i\overline{j}}} = g_{\overline{i}j} = g_{j\overline{i}} = h_{ji}$$

Also $\omega_g = \overline{\omega_g}$ so the Kähler form is real.

1.1.2 Connections

We can extend ∇ $\mathbb{C}$-linearly to $T_{\mathbb{C}}M$. Then

$$\nabla_{\frac{\partial}{\partial z_i}} \frac{\partial}{\partial z_j} = \Gamma^k_{ij}\frac{\partial}{\partial z_k} + \Gamma^{\overline{k}}_{ij}\frac{\partial}{\partial \overline{z}_k}$$
$$\nabla_{\frac{\partial}{\partial z_i}} \frac{\partial}{\partial \overline{z}_j} = \Gamma^k_{i\overline{j}}\frac{\partial}{\partial z_k} + \Gamma^{\overline{k}}_{i\overline{j}}\frac{\partial}{\partial \overline{z}_k}$$

where Γ^k_{ij} denote the Christoffel symbols. Since $\nabla J = 0$, by Equation 1 we have

$$\nabla_{\frac{\partial}{\partial z_i}}\left(J\frac{\partial}{\partial z_j}\right) = J\nabla_{\frac{\partial}{\partial z_i}}\frac{\partial}{\partial z_j} \tag{3}$$

So putting the Christoffel symbols in (3) and using that $\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_i}$ are in the i and $-i$ eigenspaces of J respectively,

$$\begin{aligned}\nabla_{\frac{\partial}{\partial z_i}}\left(J\frac{\partial}{\partial z_j}\right) &= \nabla_{\frac{\partial}{\partial z_i}}\left(i\frac{\partial}{\partial z_j}\right) \\ &= i\left(\Gamma^k_{ij}\frac{\partial}{\partial z_k} + \Gamma^{\overline{k}}_{ij}\frac{\partial}{\partial \overline{z}_k}\right)\end{aligned}$$

and

$$
\begin{aligned}
J\nabla_{\frac{\partial}{\partial z_i}}\frac{\partial}{\partial z_j} &= J\left(\Gamma_{ij}^{k}\frac{\partial}{\partial z_k}+\Gamma_{ij}^{\overline{k}}\frac{\partial}{\partial \overline{z}_k}\right)\\
&= i\left(\Gamma_{ij}^{k}\frac{\partial}{\partial z_k}-\Gamma_{ij}^{\overline{k}}\frac{\partial}{\partial \overline{z}_k}\right)\\
&\implies \Gamma_{ij}^{\overline{k}}=0
\end{aligned}
$$

By similar calculations, all Christoffel symbols are zero except Γ_{ij}^{k} and $\Gamma_{\overline{i}\overline{j}}^{\overline{k}}$. Then the connection matrix for the induced connection on $T^{1,0}M$ is θ given by

$$
\nabla\frac{\partial}{\partial z_j} = \theta_j^k\otimes\frac{\partial}{\partial z_k} = (\Gamma_{ij}^{k}dz_i)\otimes\frac{\partial}{\partial z_k} \tag{4}
$$

As M is Kähler, $d\omega_g=0$. Note that

$$
d\omega_g = 0 \iff \sum_{i,j,k}\left[\frac{\partial g_{i\overline{j}}}{\partial z_k}dz_k+\frac{\partial g_{i\overline{j}}}{\partial \overline{z}_k}d\overline{z}_k\right]\wedge dz_i\wedge d\overline{z}_j = 0 \tag{5}
$$

$$
\iff \frac{\partial g_{i\overline{j}}}{\partial z_k}=\frac{\partial g_{k\overline{j}}}{\partial z_i},\quad \frac{\partial g_{i\overline{j}}}{\partial \overline{z}_k}=\frac{\partial g_{i\overline{k}}}{\partial \overline{z}_j} \tag{6}
$$

Since $g_{ij}=g_{\overline{i}\overline{j}}=0$ by compatibility of J, we can write out the Christoffel symbols as

$$
\Gamma_{ij}^{l} = \frac{1}{2}g^{l\overline{r}}\left(\frac{\partial g_{i\overline{r}}}{\partial z_j}+\frac{\partial g_{j\overline{r}}}{\partial z_i}-\frac{\partial g_{ij}}{\partial \overline{z}_r}\right) = g^{l\overline{r}}\frac{\partial g_{i\overline{r}}}{\partial z_j} \tag{7}
$$

1.2 Chern classes

Set $n := \dim_{\mathbb{C}} M$. It is a property of the Chern connection on M that its induced curvature form on $K_M^{-1} := \wedge^n T^{1,0}M$ is equal to the trace of the curvature form on $T^{1,0}M$, that is

$$
\Theta_{K_M^{-1}} = tr(\Theta_{T^{1,0}M})
$$

Let $e_1,\ldots,e_n$ be a local frame for $T^{1,0}M$ so $e_1\wedge\ldots\wedge e_n$ is a local frame for K_M^{-1}. Since K_M^{-1} is a line bundle, the curvature form induced by the Chern connection is

$$
\Theta_{K_M^{-1}} = \partial\overline{\partial}\log h \tag{8}
$$

where $h=h(e_1\wedge\ldots\wedge e_n, e_1\wedge\ldots\wedge e_n) := \det(h_{ij}) = \det(g_{i\overline{j}})$.

M is also a Riemannian manifold. The Riemannian curvature induced by g is

$$
R_{i\overline{j}k\overline{l}} = R\left(\frac{\partial}{\partial z_i},\frac{\partial}{\partial \overline{z}_j},\frac{\partial}{\partial z_k},\frac{\partial}{\partial \overline{z}_l}\right) = g\left(R\left(\frac{\partial}{\partial z_i},\frac{\partial}{\partial \overline{z}_j}\right)\frac{\partial}{\partial \overline{z}_l},\frac{\partial}{\partial z_k}\right)
$$

and the Ricci curvature $R_{k\overline{l}} = g^{i\overline{j}}R_{i\overline{j}k\overline{l}}$. Using Equation 7 we can obtain $R_{i\overline{j}k\overline{l}}$. Recall

$$
R(u,v)w = \nabla_u\nabla_v w - \nabla_v\nabla_u w - \nabla_{[u,v]}w
$$

Thus taking $u=\frac{\partial}{\partial z_i}, v=\frac{\partial}{\partial \overline{z}_j}, w=\frac{\partial}{\partial \overline{z}_l}$ we have

$$
R\left(\frac{\partial}{\partial z_i},\frac{\partial}{\partial \overline{z}_j}\right)\frac{\partial}{\partial \overline{z}_l} = R_{i\overline{j}l}^{\overline{r}}\frac{\partial}{\partial \overline{z}_r} \tag{9}
$$

$$
= \nabla_{\frac{\partial}{\partial z_i}}\nabla_{\frac{\partial}{\partial \overline{z}_j}}\frac{\partial}{\partial \overline{z}_l} \tag{10}
$$

Using the product rule for differentiation of $g^{p\overline{r}}g_{l\overline{r}} = \delta_l^p$ we find

$$\frac{\partial g^{p\overline{r}}}{\partial z_i} = -g^{p\overline{s}}g^{n\overline{r}}\frac{\partial g_{n\overline{s}}}{\partial z_i}$$

Also from (7) above

$$\Gamma^{\overline{r}}_{\overline{j}\overline{l}} = g^{p\overline{r}}\frac{\partial g_{p\overline{j}}}{\partial \overline{z}_l}$$

Thus (10) is

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial z_i}}\left(\Gamma^{\overline{r}}_{\overline{j}\overline{l}}\frac{\partial}{\partial \overline{z}_r}\right) &= \frac{\partial}{\partial z_i}\left(g^{p\overline{r}}\frac{\partial g_{p\overline{j}}}{\partial \overline{z}_l}\right)\frac{\partial}{\partial \overline{z}_r} \\
&= \left(-g^{p\overline{s}}g^{n\overline{r}}\frac{\partial g_{n\overline{s}}}{\partial z_i}\frac{\partial g_{p\overline{j}}}{\partial \overline{z}_l} + g^{p\overline{r}}\frac{\partial^2 g_{p\overline{j}}}{\partial z_i \partial \overline{z}_l}\right)\frac{\partial}{\partial \overline{z}_r}
\end{aligned}$$

$$\begin{aligned}
\implies R_{i\overline{j}k\overline{l}} &= R^{\overline{r}}_{i\overline{j}\overline{l}}g_{\overline{r}k} \\
&= -g^{p\overline{s}}\frac{\partial g_{k\overline{s}}}{\partial z_i}\frac{\partial g_{p\overline{j}}}{\partial \overline{z}_l} + \frac{\partial^2 g_{k\overline{j}}}{\partial z_i \partial \overline{z}_l}
\end{aligned}$$

The Ricci curvature is

$$R_{i\overline{j}} = -\frac{\partial^2}{\partial z_i \partial \overline{z}_j}(\log\det g_{k\overline{l}})$$

The Ricci form is defined to be

$$Ric(g) = \frac{i}{2}\sum_{i,j} R_{i\overline{j}}dz_i \wedge d\overline{z}_j = -\frac{i}{2}\partial\overline{\partial}\log\det g_{i\overline{j}}$$

Note that the Ricci form is, up to a factor of $-\frac{i}{2}$, the same as $\Theta_{K_M^{-1}}$ in (8). On a Kähler manifold, the Chern connection and Levi-Civita connection are equivalent on $T^{1,0}M \cong TM$ [Huy05, Prop 4.A.9]. Let $\Omega_i^j = g^{j\overline{p}}R_{i\overline{p}k\overline{l}}dz_k \wedge d\overline{z}_l$. By the equivalence between Chern and Levi-Civita connections, we can define the Chern classes in terms of Ω.

Let $c(M) = \det\left(I + t\frac{i}{2\pi}\Omega\right)$. It is a fact that the coefficients on t^k are real closed (k,k) forms and their cohomology classes in $H^{k,k}(M,\mathbb{C}) \cap H^{2k}(M,\mathbb{R})$ are independent of g. [Huy05, pgs 194–195, 198]. The kth Chern class $c_k(M)$ is defined to be the cohomology class represented by this coefficient.

Definition 7. The <u>first Chern class</u> $c_1(M)$ is $\frac{i}{2\pi}[tr(\Omega)] = \frac{1}{\pi}[Ric(g)]$.

We write $c_1(M) > 0$ if the first Chern class can be represented by a form with coefficients in local coordinates given by $\sqrt{-1}\cdot\phi_{i\overline{j}}$ for $\phi_{i\overline{j}}$ positive definite, $c_1(M) < 0$ if $-c_1(M) > 0$ and $c_1(M) = 0$ if it can be represented by a form cohomologous to zero.

Definition 8. $[\omega] \in H^{1,1}(M,\mathbb{C}) \cap H^2(M,\mathbb{R})$ is a <u>Kähler class</u> if it can be represented by a form corresponding to a Kähler metric, i.e. we can choose ω to be $\frac{i}{2}\sum_{i,j} g_{i\overline{j}}dz_i \wedge d\overline{z}_j$ for some $g_{i\overline{j}}$ positive definite.

Remark 9. In particular if $c_1(M) > 0$ for a compact complex manifold M, then $c_1(M)$ is a Kähler class since it can be represented by a positive definite closed $(1,1)$ form, so M is a Kähler manifold. In general if L is a positive line bundle over M, i.e. a holomorphic line bundle with $c_1(L) > 0$, then M is Kähler. The case $c_1(M) > 0$ is the special case $L = K_M^{-1}$.

2 Calabi-Yau Theorem

Definition 10. A Kähler metric g on a Kähler manifold M is said to be Kähler-Einstein if

$$Ric(g) = \lambda\omega_g$$

for some $\lambda \in \mathbb{R}$. In this case, M is called a Kähler-Einstein manifold.

The question of when a complex manifold M admits a Kähler-Einstein metric has been answered in the cases $c_1(M) \leq 0$. The answer makes use of the following theorem first conjectured by Calabi and later proved by Yau.

Theorem 11 (Calabi-Yau). *Let M be a compact Kähler manifold. Let Ω be a representative form for $\pi c_1(M)$ and $[\omega] \in H^{1,1}(M,\mathbb{C}) \cap H^2(M,\mathbb{R})$ a Kähler class. Then there exists a unique Kähler metric g with $\omega_g \in [\omega]$ such that $Ric(g) = \Omega$.*

Thus when $c_1(M) = 0$, the Calabi-Yau theorem implies M has a Ricci flat metric g, i.e. $Ric(g) = 0$, so g is a Kähler-Einstein metric. Aubin and Yau independently proved that when $c_1(M) < 0$, there exists a unique Kähler-Einstein metric g such that $Ric(g) = -\omega_g$. The full answer is in progress when $c_1(M) > 0$. In this case, M is called a Fano manifold.

Some ideas in the proof of Theorem 11. I will not give a proof of the Calabi-Yau theorem but will note a few points from the proof in [Tia00, Theorem 5.1].

The proof makes use of the $\partial\overline{\partial}$-lemma which will also be used later. The part of the lemma needed is the following.

Lemma 12 ($\partial\overline{\partial}$-Lemma). *Let (M,g) be a compact Kähler manifold. Suppose $\alpha \in H^{1,1}(M,\mathbb{C})$ is d-exact. Then there exists a smooth function β such that $\alpha = \partial\overline{\partial}\beta$.*

Remark 13. If ω_1 and ω_2 are two cohomologous Kähler forms associated to Kähler metrics (in particular they are real), then by the $\partial\overline{\partial}$-lemma $\omega_1 - \omega_2 = \partial\overline{\partial}\beta$ is real so

$$\partial\overline{\partial}\beta = \overline{\partial\overline{\partial}\beta} = \overline{\partial}\partial\overline{\beta} = \partial\overline{\partial}(-\overline{\beta})$$

which implies we can choose β such that $\beta = i \cdot f$ for some $f \in C^\infty(M,\mathbb{R})$.

Proof of Lemma, [Huy05, Cor 3.2.10]. Since α is d-exact we can write $\alpha = d\eta$ for some $\eta \in H^1(M,\mathbb{C})$. M is Kähler so the notions of ∂, $\overline{\partial}$ and d harmonicity are equivalent. If $(\cdot,\cdot)$ denotes the inner product on (p,q) forms given by

$$(\psi,\eta) \mapsto \int_M g_{\mathbb{C}}(\psi,\overline{\eta})\omega_g^n = \int_M h(\psi,\eta)\omega_g^n$$

then d^* is the formal adjoint of d with respect to this inner product. ψ is d-harmonic if and only if $d\psi = d^*\psi = 0$. Thus for all $\psi \in \mathcal{H}^{1,1}(M,\mathbb{C})$, the space of harmonic $(1,1)$ forms,

$$(\alpha,\psi) = (d\eta,\psi) = (\eta, d^*\psi) = 0$$

i.e. $\alpha \perp \mathcal{H}^{1,1}(M,\mathbb{C})$. We know $d\alpha = 0$, thus $\partial\alpha = \overline{\partial}\alpha = 0$. Since $\alpha \in \ker\partial$, by the Hodge decomposition for ∂, α is in the direct sum of the harmonic $(1,1)$ forms and the image of ∂ on $(0,1)$ forms. We know $\alpha \notin \mathcal{H}^{1,1}(M,\mathbb{C})$ hence $\alpha = \partial\gamma$ for some $(0,1)$ form γ. Again by Hodge decomposition, now for $\overline{\partial}$

$$\begin{aligned} &\gamma = \overline{\partial}\beta + \overline{\partial}^*\beta' + \beta'' \\ \implies &\alpha = \partial\overline{\partial}\beta + \partial\overline{\partial}^*\beta' = -\overline{\partial}\partial\beta - \overline{\partial}^*\partial\beta' \end{aligned}$$

for some $\beta'' \in \mathcal{H}^{0,1}(M)$. Note $\overline{\partial}^*\partial = -\partial\overline{\partial}^*$ by the Hodge identities. So

$$\begin{aligned}
\overline{\partial}\alpha = 0 &\implies \overline{\partial\partial}^*\partial\beta' = 0 \\
&\implies 0 = (\overline{\partial\partial}^*\partial\beta', \partial\beta') = (\overline{\partial}^*\partial\beta', \overline{\partial}^*\partial\beta') = ||\overline{\partial}^*\partial\beta'||^2 \\
&\implies \overline{\partial}^*\partial\beta' = 0 \\
&\implies \alpha = \partial\overline{\partial}\beta
\end{aligned}$$

□

Returning to the Calabi-Yau theorem, in local coordinates

$$\omega_g = \frac{i}{2}\sum_{i,j} g_{i\bar{j}} dz_i \wedge d\overline{z}_j$$
$$Ric(g) = -\frac{i}{2}\partial\overline{\partial}\log\det(g_{i\bar{j}})$$

Since $Ric(g)$ and Ω are cohomologous, the $\partial\overline{\partial}$-lemma says there exists a real smooth function f such that

$$\Omega - Ric(g) = \frac{i}{2}\partial\overline{\partial}f$$

We normalize f so that $\int_M e^{-f}\omega_g^n = \int_M \omega_g^n$, and then such an f is unique. We want to find some metric $\omega \in [\omega_g]$ such that $Ric(\omega) = \Omega$. Again by the $\partial\overline{\partial}$-lemma, ω must be of the form $\omega_g + \frac{i}{2}\partial\overline{\partial}\phi$, so ω corresponds to the metric with coefficients $g_{i\bar{j}} + \frac{\partial^2\phi}{\partial z_i \partial \overline{z}_j}$. Then

$$Ric(\omega_g + \frac{i}{2}\partial\overline{\partial}\phi) = \Omega = Ric(g) + \frac{i}{2}\partial\overline{\partial}f$$

In local coordinates this is

$$\begin{aligned}
-\frac{i}{2}\partial\overline{\partial}\log\det\left(g_{i\bar{j}} + \frac{\partial^2\phi}{\partial z_i\partial\overline{z}_j}\right) &= -\frac{i}{2}\partial\overline{\partial}\log(\det(g_{i\bar{j}})) + \frac{i}{2}\partial\overline{\partial}f \\
\implies -\partial\overline{\partial}f &= \partial\overline{\partial}\log\left(\frac{\det\left(g_{i\bar{j}} + \frac{\partial^2\phi}{\partial z_i\partial\overline{z}_j}\right)}{\det(g_{i\bar{j}})}\right) \\
\implies -f + c &= \log\left(\frac{\det\left(g_{i\bar{j}} + \frac{\partial^2\phi}{\partial z_i\partial\overline{z}_j}\right)}{\det(g_{i\bar{j}})}\right)
\end{aligned}$$

for some constant c, since harmonic functions on a compact complex manifold are constant. The left hand side is defined globally so the right hand side is as well. Exponentiating both sides gives

$$\det\left(g_{i\bar{j}} + \frac{\partial^2\phi}{\partial z_i\partial\overline{z}_j}\right) = e^{-f+c}\det(g_{i\bar{j}}) \tag{11}$$

Equation (11) is equivalent to

$$(\omega_g + \frac{i}{2}\partial\overline{\partial}\phi)^n = e^{-f+c}\omega_g^n \tag{12}$$

Note that using Stokes' theorem for $0 < m \leq n$

$$\begin{aligned}\int_M \omega_g^{n-m} \wedge (\partial\overline{\partial}\phi)^m &= \int_M \partial(\omega_g^{n-m} \wedge (\partial\overline{\partial}\phi)^{m-1} \wedge \overline{\partial}\phi) \\ &= \int_M d(\omega_g^{n-m} \wedge (\partial\overline{\partial}\phi)^{m-1} \wedge \overline{\partial}\phi) \\ &= 0\end{aligned}$$

since ω_g is closed and we can replace ∂ with d since the form is a $(n-1,n)$ form. So integrating both sides of Equation 12 over M implies

$$\int_M \omega_g^n = \int_M e^{-f+c}\omega_g^n$$

so by the normalization condition above, $c = 0$. Thus to find an ω as in the Calabi-Yau theorem, we need to solve the complex Monge-Ampère equation, which is

$$(\omega_g + \frac{i}{2}\partial\overline{\partial}\phi)^n = e^{-f}\omega_g^n \tag{13}$$

Yau's proof of this conjecture is given in [Tia00, Theorem 5.1] and involves a continuity argument on solutions to

$$(\omega_g + \frac{i}{2}\partial\overline{\partial}\phi)^n = e^{-f_s}\omega_g^n \tag{14}$$

where $f_s := sf + c_s$, $s \in [0,1]$, and c_s are constants uniquely determined by the normalization condition $\int_M (e^{-f_s} - 1)\omega_g^n = 0$. It is shown that the set

$$S = \{s \in [0,1] | \text{ there is a solution to (14) for all } t \leq s\}$$

is open and closed (and non-empty, since it contains zero by setting ϕ = constant) hence $S = [0,1]$ and there is a solution at f.

3 Calabi-Futaki invariant

There are obstructions to the existence of Kähler-Einstein metrics on Fano manifolds. The vanishing of the Calabi-Futaki invariant is necessary and is an obstruction related to holomorphic vector fields.

Assume M is a compact Fano manifold. Let $Ka(M)$ denote the set of Kähler classes on M and $\eta(M)$ the space of holomorphic vector fields on M, i.e. in local coords $z_1, \ldots, z_n$, vector fields of the form $X_i \frac{\partial}{\partial z_i}$ with X_i holomorphic.

Choose $[\omega] \in Ka(M)$ and Kähler metric $\omega_g \in [\omega]$. Let $s(g)$ denote the complex scalar curvature of g, i.e. $s(g) = g^{i\bar{j}}R_{i\bar{j}}$ locally. Define a function h_g on M by

$$s(g) - \frac{1}{V}\int_M s(g)\omega_g^n = \Delta h_g$$

where $V = \int_M \omega_g^n$ is the volume.

Definition 14. The Calabi-Futaki invariant f_M is

$$f_M : Ka(M) \times \eta(M) \to \mathbb{C}$$

$$f_M([\omega_g], X) = \int_M X(h_g)\omega_g^n$$

Calabi and Futaki proved

Theorem 15. *$f_M([\omega], X)$ is a holomorphic invariant independent of the representative g chosen in $[\omega]$. In particular, if there exists a constant scalar curvature metric $\omega_g \in [\omega]$, then $f_M([\omega], -) = 0$.*

We can restrict the first argument to $\pi c_1(M) > 0$. Let $\pi c_1(M) = [\omega_g] = [Ric(\omega_g)]$. So $Ric(\omega_g) - \omega_g = \frac{i}{2}\partial\overline{\partial}h_g$ for some function h_g, by the $\partial\overline{\partial}$-Lemma. A Kähler manifold locally admits normal coordinates about any $x \in M$ [Tia00, Prop 1.6], where $g_{i\bar{j}}(x) = \delta_{i\bar{j}}$. We can assume $\omega_g = \frac{i}{2}\sum dz_i \wedge d\bar{z}_i$, $Ric(g) = \frac{i}{2}\sum R_{i\bar{i}} dz_i \wedge d\bar{z}_i$ at x, and

$$\begin{aligned} Ric(g) \wedge \omega_g^{n-1} &= (n-1)! \sum_i R_{i\bar{i}} \frac{\omega_g^n}{n!} \\ &= \frac{1}{n} s(g)\omega_g^n \end{aligned}$$

at x. Thus

$$\begin{aligned} \frac{1}{V}\int_M s(g)\omega_g^n &= \frac{n}{V}\int_M Ric(g) \wedge \omega_g^{n-1} \\ &= \frac{n}{V}\int_M \left(\omega_g + \frac{i}{2}\partial\overline{\partial}h_g\right) \wedge \omega_g^{n-1} \\ &= \frac{n}{V}\int_M \omega_g^n = n \end{aligned}$$

where the last step follows since $\int_M \partial\overline{\partial}h_g \wedge \omega_g^{n-1} = \int_M \partial(\overline{\partial}h_g \wedge \omega_g^{n-1})$ as ω_g is closed, which equals $\int_M d(\overline{\partial}h_g \wedge \omega_g^{n-1})$ as $\overline{\partial}h_g \wedge \omega_g^{n-1}$ is an $(n-1, n)$ form, so this integral vanishes by Stokes' theorem.

So in the special case $[\omega_g] = \pi c_1(M)$, $f_M([\omega_g], -)$ gives Futaki's invariant [Fut88, §3.1] where h_g is equivalently defined as

$$Ric(\omega_g) - \omega_g = \frac{i}{2}\partial\overline{\partial}h_g \tag{15}$$

This is equivalent since contracting the coefficients in (15) with $g^{i\bar{j}}$ gives $s(g) - n = \Delta h_g$ and we already saw that $n = \frac{1}{V}\int_M s(g)\omega_g^n$.

Remark 16. In $\pi c_1(M) > 0$, the notions of constant scalar curvature Kähler (cscK) metrics and KE metrics are equivalent. We have $\pi c_1(M) = [\omega_g] = [Ric(g)]$ and $Ric(g) - \omega_g = \frac{i}{2}\partial\overline{\partial}h_g$. KE implies cscK since $Ric(g) = \lambda\omega_g$ implies locally $R_{i\bar{j}} = \lambda g_{i\bar{j}}$ therefore $s(g) = n\lambda$ by taking the trace of both sides, where $n = \dim_{\mathbb{C}} M$.

Conversely, suppose $\omega_g \in \pi c_1(M)$ has constant scalar curvature. We know $n = \frac{1}{V}\int_M s(g)\omega_g^n$. Thus if s is constant, $s = n$ so h_g is harmonic on a compact manifold hence constant. Therefore $Ric(g) = \omega_g$ and (M, g) is Kähler-Einstein.

Proof of Theorem 15, [Fut83],[TD92]. The following proof is for the Kähler-Einstein case using the Futaki invariant and follows [Fut83] and [TD92]. I will give the idea behind the proof for the general case, given in [Tia00, Theorem 3.3]. The second statement of the theorem is clear in the more general case; if $s(g)$ is constant then $\Delta h_g = 0$ so h_g is constant. Hence $X(h_g) = 0$ $\forall X \in \eta(M)$ and $f_M([\omega], -) = 0$.

Let $f_M(X) := f_M(\pi c_1(M), X)$. To show $f_M(X)$ is independent of the representative g chosen, it suffices to show $f_t(X) := \int_M X(h_{g_t})\omega_t^n$ is locally constant for an arbitrary differentiable family of Kähler metrics $\omega_t := \omega_{g_t}$ in $\pi c_1(M)$, that is $\frac{d}{dt} f_t(X) = 0$. This will suffice since the set of all Kähler metrics in $\pi c_1(M)$ is a cone, hence contractible.

Let $\pi c_1(M)$ be represented by ω. By the $\partial\overline{\partial}$-lemma, there exists smooth real functions ψ_t such that

$$\omega_t - \omega = \frac{i}{2}\partial\overline{\partial}\psi_t$$

Then differentiating with respect to t gives

$$\frac{\partial \omega_t}{\partial t} = \frac{i}{2}\partial\overline{\partial}\left(\frac{\partial \psi_t}{\partial t}\right)$$

Set $\phi_t := \frac{\partial \psi_t}{\partial t}$. Here ω_t^n denotes $\det(g_t)_{i\bar{j}}$. Then using $Ric(\omega_t) - \omega_t = \frac{i}{2}\partial\overline{\partial}h_t$

$$\begin{aligned}
\frac{\partial}{\partial t}(\omega_t^n) &= n\frac{\partial \omega_t}{\partial t} \wedge \omega_t^{n-1} \\
&= \frac{ni}{2}\partial\overline{\partial}\phi_t \wedge \omega_t^{n-1} \\
&= \Delta\phi_t \omega_t^n
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial t}(Ric(\omega_t)) &= -\frac{i}{2}\partial\overline{\partial}\left(\frac{\partial}{\partial t}\log \omega_t^n\right) \\
&= -\frac{i}{2}\partial\overline{\partial}\left(\frac{\Delta\phi_t\omega_t^n}{\omega_t^n}\right) \\
&= -\frac{i}{2}\partial\overline{\partial}\Delta\phi_t
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial t}(\partial\overline{\partial}h_{g_t}) &= -2i\frac{\partial}{\partial t}(Ric(\omega_t) - \omega_t) \\
&= \partial\overline{\partial}(-\Delta\phi_t - \phi_t)
\end{aligned}$$

So choose $h_{g_t} \equiv h_t$ such that

$$\frac{\partial h_t}{\partial t} = -\Delta\phi_t - \phi_t$$

Then

$$\begin{aligned}
\frac{d}{dt}f_t(X) &= \int_M \frac{\partial}{\partial t}(X(h_t)\omega_t^n) \\
&= \int_M \left(X\left(\frac{\partial h_t}{\partial t}\right) + X(h_t)\Delta\phi_t\right)\omega_t^n \\
&= \int_M (X(-\Delta\phi_t - \phi_t) + X(h_t)\Delta\phi_t)\omega_t^n
\end{aligned}$$

The following argument is from [TD92]. Note that

$$\begin{aligned}
X(\Delta\phi_t) &= X^i\frac{\partial}{\partial z_i}\left({g_t}^{j\bar{k}}\frac{\partial^2\phi_t}{\partial z_j\partial\overline{z}_k}\right) \\
&= \frac{\partial}{\partial z_i}\left(X^i {g_t}^{j\bar{k}}\frac{\partial^2\phi_t}{\partial z_j\partial\overline{z}_k}\right) - \Delta\phi_t\frac{\partial X^i}{\partial z_i}
\end{aligned}$$

The first term vanishes when integrating over M, by the divergence theorem, and the second term is $-div(X) \cdot \Delta\phi_t$. Similarly we can replace $X(-\phi_t)$ with $div(X) \cdot \phi_t$. Also in local coordinates, $i_X\omega_g = \frac{i}{2}X^j g_{j\bar{k}} d\overline{z_k}$ and ω_g is closed so $\frac{\partial g_{i\bar{j}}}{\partial \overline{z_k}} = \frac{\partial g_{i\bar{k}}}{\partial \overline{z_j}}$. Since the X^j are holomorphic

$$
\begin{aligned}
-2i\overline{\partial}(i_X\omega_t) &= X^j\overline{\partial}(g_{t_{j\bar{k}}})d\overline{z_k} \\
&= X^j\left[\sum_{i<k}\frac{\partial g_{t_{j\bar{k}}}}{\partial \overline{z_i}}d\overline{z_i}\wedge d\overline{z_k} + \sum_{i>k}\frac{\partial g_{t_{j\bar{k}}}}{\partial \overline{z_i}}d\overline{z_i}\wedge d\overline{z_k}\right] \\
&= X^j\left[\sum_{i<k}\frac{\partial g_{t_{j\bar{k}}}}{\partial \overline{z_i}}d\overline{z_i}\wedge d\overline{z_k} - \sum_{i<k}\frac{\partial g_{t_{j\bar{k}}}}{\partial \overline{z_i}}d\overline{z_i}\wedge d\overline{z_k}\right] = 0
\end{aligned}
$$

So by the Hodge theorem

$$\iota_X\omega_t = \alpha_t + \overline{\partial}\eta_t$$

for some harmonic $(0,1)$ form α_t and smooth function η_t. The α_t will vanish in the integral (n.b. $(\alpha_t, \Delta\phi_t) = (\Delta\alpha_t, \phi_t) = 0$) so we can assume it is zero. Then $L_X\omega_t = d(\iota_X\omega_t) = \partial(\iota_X\omega_t)$ by Cartan's formula and $(divX)\omega_t^n = L_X(\omega_t^n)$, so

$$\partial(\iota_X\omega_t) = \partial\overline{\partial}\eta_t \implies div(X) = \Delta\eta_t$$

Since $(\phi_t, \Delta\eta_t) = (\Delta\phi_t, \eta_t)$ with respect to the inner product (,) induced by integrating over M

$$
\begin{aligned}
\frac{d}{dt}f_t(X) &= \int_M \left[(div(X) + X(h_t))\Delta\phi_t + div(X)\phi_t\right]\omega_t^n \\
&= \int_M (\Delta\eta_t + X(h_t) + \eta_t)\Delta\phi_t\omega_t^n
\end{aligned}
$$

We show $\overline{\partial}(\Delta\eta_t + X(h_t) + \eta_t) = 0$. Since $\frac{\partial^2 h_t}{\partial z_i \partial \overline{z}_j} = R_{i\bar{j}} - g_{t i\bar{j}}$ (where $R_{i\bar{j}}$ depends on t)

$$
\begin{aligned}
\overline{\partial}(X(h_t)) &= \overline{\partial}\left(X^i\frac{\partial h_t}{\partial z_i}\right) \\
&= X^i\frac{\partial^2 h_t}{\partial z_i\partial\overline{z}_j}d\overline{z}_j \\
&= \iota_X(Ric(g_t) - \omega_t) \\
\overline{\partial}\eta_t &= \iota_X\omega_t
\end{aligned}
$$

and from the definition of $Ric(g_t)$ we show $\overline{\partial}\Delta\eta_t = -\iota_X Ric(g_t)$ (from [Tia00, pg 25])

$$
\begin{aligned}
\iota_X Ric(g_t) &= -\frac{i}{2}X^i\frac{\partial^2}{\partial z_i\partial\overline{z}_j}\log\det(g_{t_{k\bar{l}}})d\overline{z}_j \\
&= -\frac{i}{2}\overline{\partial}\left(X^i\frac{\partial}{\partial z_i}\log\det(g_{t_{k\bar{l}}})\right) \\
&= -\frac{i}{2}\overline{\partial}\left(X^i g_t^{k\bar{l}}\frac{\partial g_{t_{k\bar{l}}}}{\partial z_i}\right) \\
&= -\frac{i}{2}\overline{\partial}\left(X^i g_t^{k\bar{l}}\frac{\partial g_{t_{i\bar{l}}}}{\partial z_k}\right) \\
&= -\frac{i}{2}\overline{\partial}\left(g_t^{k\bar{l}}\frac{\partial}{\partial z_k}(X^i g_{t_{i\bar{l}}}) - g_t^{k\bar{l}}g_{t_{i\bar{l}}}\frac{\partial X^i}{\partial z_k}\right) \\
&= -\frac{i}{2}\overline{\partial}\left(g_t^{k\bar{l}}\frac{\partial}{\partial z_k}(X^i g_{t_{i\bar{l}}})\right) \\
&= -\overline{\partial}\left(g_t^{k\bar{l}}\frac{\partial}{\partial z_k}\frac{\partial}{\partial\overline{z}_l}\eta_t\right) \\
&= -\overline{\partial}\Delta\eta_t
\end{aligned}
$$

The fact that X is holomorphic is used in the second line and sixth lines, the definition of the inverse of a matrix A as $\frac{1}{\det A} adj(A)$, where $adj(A)$ is the adjugate matrix, is used in the third line, the fourth line uses that ω_g is closed, the fifth line is the Chain rule, and the penultimate line uses $\iota_X\omega_t = \overline{\partial}\eta_t$. So $\overline{\partial}(\Delta\eta_t + \eta_t + X(h_t)) = 0$ and by the chain rule and divergence theorem $\frac{d}{dt} f_t(X) = 0$.

In the more general case, one can show that $f_M([\omega], X) = \int_M \theta_X \Delta_g h_g \omega_g^n$, for a specified function θ_X. By the Hodge theorem, $i_X\omega_g = \frac{i}{2}(\alpha + \overline{\partial}\theta_X)$ for some harmonic 1-form α and smooth function θ_X. Thus $X^j = g^{j\overline{k}}\left(\alpha_{\overline{k}} + \frac{\partial\theta_X}{\partial\overline{z_k}}\right)$.

Let Δ_g denote the $\overline{\partial}$-Laplacian on functions, which is $\overline{\partial}^*\overline{\partial}$. Note that the inner product $g_{\mathbb{C}}(\psi, \overline{\eta})$ on forms is the dual of the inner product on vector fields, so has coefficients $g^{j\overline{k}}$. Then

$$\begin{aligned} f_M([\omega], X) &= \int_M X(h_g)\omega_g^n \\ &= \int_M X^j \frac{\partial h_g}{\partial z_j}\omega_g^n \\ &= \int_M g^{j\overline{k}}\left(\alpha_{\overline{k}} + \frac{\partial\theta_X}{\partial\overline{z_k}}\right)\frac{\partial h_g}{\partial z_j}\omega_g^n \\ &= (\partial h_g, \overline{\alpha}) + (\partial h_g, \partial\theta_X) \\ &= (\alpha, \overline{\partial}h_g) + (\overline{\partial}\theta_X, \overline{\partial}h_g) \\ &= (\overline{\partial}^*\alpha, h_g) + (\theta_X, \overline{\partial}^*\overline{\partial}h_g) \\ &= \int_M \theta_X\Delta_g h_g\omega_g^n \end{aligned}$$

The final line follows since α is harmonic, so $\overline{\partial}^*\alpha = 0$. Then defining $F(g, X) = (n+1)2^{n+1}\int_M h_g\Delta\theta_X\omega_g^n$, one takes a family of metrics $\{g_t\}$ in the given Kähler class $[\omega]$ and shows $\frac{d}{dt}F(g_t, X)|_{t=0} = 0$, done in [Tia00, pg 24–27]. □

3.1 Formula for the Calabi-Futaki invariant

With the additional condition that $X \in Vect(M)$ be non-degenerate, Futaki gives a formula for $f_M(\pi c_1(M), -)$, by looking at the zero set of X, [Fut88].

Here is the set-up. Suppose $Z \subseteq M$ is a smooth complex submanifold and let $N_{M|Z}$ denote the normal bundle TM/TZ to Z in M. A metric g on M induces an orthogonal decomposition $TM|_Z = TZ \oplus N_{M|Z}$. If ∇ is the Levi-Civita connection on M, ∇X induces a section DX of $End(N_{M|Z})$, given by restricting to vectors in $N_{M|Z}$, taking the covariant derivative of X in the given direction, and projecting the result to $N_{M|Z}$. With the orthogonal decomposition above, $DX = (\nabla X)^\perp|_{N_{M|Z}}$ where $(\nabla X)^\perp$ denotes the component in TM perpendicular to TZ.

Definition 17. We say $X \in Vect(M)$ is non-degenerate if

$$zero(X) = \coprod_{\lambda\in\Lambda} Z_\lambda$$

where the Z_λ are smooth complex connected submanifolds, and

$$D_zX : T_zM/T_zZ_\lambda \to T_zM/T_zZ_\lambda$$

is a non-degenerate linear map of vector spaces, i.e. has nonzero determinant, for all $z \in Z_\lambda$, $\forall \lambda \in \Lambda$.

Then

Theorem 18 ([Fut88, Theorem 5.2.8]). *For X a non-degenerate vector field on M*

$$f_M(\pi c_1(M), X) = \frac{\pi^n}{n+1} \sum_{\lambda \in \Lambda} \int_{Z_\lambda} \frac{[tr(L_\lambda(X)) + c_1(M)]^{n+1}}{\det(L_\lambda(X) + \frac{i}{2\pi} K_\lambda)} \tag{16}$$

where $n = \dim_{\mathbb{C}} M$, $L_\lambda(X) = (\nabla X)^\perp|_{N_{M|Z_\lambda}}$ and K_λ is the induced curvature form on $N_{M|Z_\lambda}$.

Consider the special case where M is a complex surface and $\Lambda = \Lambda_0 \cup \Lambda_1$, where Λ_i consists of i dimensional submanifolds. Since only forms of degree $2 \cdot \dim Z_\lambda$ contribute to the integral in (16), the sum over $\lambda \in \Lambda_0$ becomes

$$\frac{\pi^2}{3} \sum_{\lambda \in \Lambda_0} \frac{tr(L_\lambda(X))^3}{\det(L_\lambda(X))}$$

For one-dimensional submanifolds Z_λ, TZ_λ and the normal bundle are line bundles, i.e. rank one vector bundles, so $L_\lambda(X)$ and K_λ are both one-by-one matrices hence we can omit the trace and determinant. Note that X is non-degenerate so $L_\lambda(X) \neq 0$. Then using an expansion for the denominator and omitting terms not of degree 2 we find

$$\begin{aligned}
&\int_{Z_\lambda} \frac{(L_\lambda(X) + c_1(M))^3}{L_\lambda(X) + \frac{i}{2\pi} K_\lambda} \\
&= \int_{Z_\lambda} \frac{L_\lambda(X)^3 + 3L_\lambda(X)^2 c_1(M)}{L_\lambda(X)(1 + \frac{i}{2\pi} K_\lambda L_\lambda(X)^{-1})} \\
&= \int_{Z_\lambda} \left((L_\lambda(X)^2 + 3L_\lambda(X) c_1(M))(1 - \frac{i}{2\pi} K_\lambda L_\lambda(X)^{-1}) \right) \\
&= \int_{Z_\lambda} \left(3L_\lambda(X) c_1(M) - \frac{i}{2\pi} L_\lambda(X) K_\lambda \right) \\
&= L_\lambda(X)(2c_1(M)(Z_\lambda) + 2 - 2g(Z_\lambda))
\end{aligned}$$

The final step was obtained as follows. K_λ is the induced curvature form on the line bundle $N_{M|Z_\lambda}$ thus $[\frac{i}{2\pi} K_\lambda] = c_1(N_{M|Z_\lambda})$. Note that

$$TM|_{Z_\lambda} = TZ_\lambda \oplus N_{M|Z_\lambda} \implies c_1(TM|_{Z_\lambda}) = c_1(TZ_\lambda) + c_1(N_{M|Z_\lambda}) \tag{17}$$

Then using the pairing of cohomology on homology, given by integrating a form over a submanifold, we find by the Gauss-Bonnet theorem

$$\int_{Z_\lambda} c_1(Z_\lambda) = \int_{Z_\lambda} c_1(TZ_\lambda) = \int_{Z_\lambda} \frac{C_\lambda}{2\pi} \Phi = \chi(Z_\lambda)$$

where Φ is the volume form and C_λ is the Gaussian curvature of Z_λ, using the fact that $C_\lambda \Phi$ is i times the curvature form on the line bundle TZ_λ, see [GH94, pg 77]. Further

$$\begin{aligned}
c_1(TM|_{Z_\lambda})(Z_\lambda) &= \int_{Z_\lambda} c_1(TM|_{Z_\lambda}) = \int_{Z_\lambda} c_1(TM) \\
&= \int_{Z_\lambda} c_1(M) \\
&= c_1(M)(Z_\lambda)
\end{aligned}$$

So by Equation 17

$$\begin{aligned}
c_1(N_{M|Z_\lambda})(Z_\lambda) &= c_1(TM|_{Z_\lambda})(Z_\lambda) - c_1(TZ_\lambda)(Z_\lambda) \\
\therefore \int_{Z_\lambda} \frac{i}{2\pi} K_\lambda &= c_1(N_{M|Z_\lambda})(Z_\lambda) \\
&= c_1(M)(Z_\lambda) - \chi(Z_\lambda) \\
&= c_1(M)(Z_\lambda) - (2 - 2g(Z_\lambda))
\end{aligned}$$

and we have the result above. So

$$f_M(\pi c_1(M), X) = \frac{\pi^2}{3} \sum_{\lambda \in \Lambda_0} \frac{tr(L_\lambda(X))^3}{\det(L_\lambda(X))} + \frac{\pi^2}{3} \sum_{\lambda \in \Lambda_1} L_\lambda(X)(2c_1(M)(Z_\lambda) + 2 - 2g(Z_\lambda))$$

3.2 Example 1

This example was done in [Tia00, pg 32–33]. Consider $\mathbb{CP}^2(p)$, the blow up of $\mathbb{CP}^2$ in a point p. Using the automorphism group of $\mathbb{CP}^2$, $SL(3, \mathbb{C})/\sim$ where $A \sim \lambda A$, $\forall \lambda \in \mathbb{C}^*$, we may assume $p = [1:0:0]$.

The blow up of a general complex manifold in a point p is given by taking a local coordinate chart centred about p, blowing up at the origin in $\mathbb{C}^n$ and then glueing the resulting $B_0\mathbb{C}^n$ back onto the manifold. The following theory is from [GH94, pg 182–185].

Here we consider a neighbourhood $U = \{[x:y:z] | x \neq 0\} \cong \mathbb{C}^2$ in $\mathbb{CP}^2$. We have a parametrization given by

$$f : \mathbb{C}^2 \to U$$

$$(x, y) \mapsto [1 : x : y]$$

and $B_0\mathbb{C}^2 = \{((x,y),[\xi:\eta]) | x\eta = y\xi\} \subset \mathbb{C}^2 \times \mathbb{CP}^1$. Thus there is a commutative diagram

$$\begin{tikzcd}
B_0\mathbb{C}^2 \arrow[d, "\pi_1"] \arrow[dr] & \\
\mathbb{C}^2 \arrow[r, "f"'] & U
\end{tikzcd}$$

where π_1 is the projection onto the first factor. Note that $f\pi_1$ restricts to an isomorphism $B_0\mathbb{C}^2 - \{(0,0) \times \mathbb{CP}^1\} \to U - \{p\}$. Then the blow-up of $\mathbb{CP}^2$ at p is obtained by glueing along this restriction

$$\begin{aligned}
M &= \mathbb{CP}^2 \backslash \{p\} \cup_{f\pi_1} B_0\mathbb{C}^2 \\
&= (\mathbb{CP}^2 \backslash \{p\}) \cup \tilde{U}
\end{aligned}$$

where $\tilde{U} = \{([1:x:y],[\xi:\eta]) | x\eta = y\xi\}$ and $E = \{([1:0:0],[\xi:\eta])\} \cong \mathbb{CP}^1$ is the exceptional divisor which has replaced p. Points $([1:x:y],[\xi:\eta]) \in \tilde{U} \backslash E$ are identified one-to-one with points $[1:x:y] \in \mathbb{CP}^2 \backslash \{p\}$. Away from E, M is isomorphic to $\mathbb{CP}^2$, which is a complex manifold. We define local coordinates about E as follows. Take a cover V_0, V_1 of $\tilde{U}$ given by

$$V_0 = \{([1:x:y],[\xi:\eta]) \in \tilde{U} | \xi \neq 0\}$$

$$V_1 = \{([1:x:y],[\xi:\eta]) \in \tilde{U} | \eta \neq 0\}$$

On these open sets we have coordinates $(z_1^0, z_2^0) = (x, \eta/\xi)$ and $(z_1^1, z_2^1) = (\xi/\eta, y)$ since y and x are then respectively determined by $x\eta = y\xi$. In particular $E \cap V_0 = \{z_1^0 = 0\}, E \cap V_1 = \{z_2^1 = 0\}$. Thus the transition functions for $[E]|_E \cong N_{M|E}$ are the inverse of those for the hyperplane bundle. Any line bundle L over $\mathbb{CP}^1$ is a multiple of the hyperplane bundle. This multiple is denoted $\deg(L)$ and corresponds to the image of L under the isomorphism

$$\{\text{line bundles}\} = H^1(\mathbb{CP}^1, \mathcal{O}^*_{\mathbb{CP}^1}) \cong H^2(\mathbb{CP}^1, \mathbb{Z}) \cong \mathbb{Z}$$
$$L \leftrightarrow \deg(L)$$

N.B. This isomorphism arises from taking the long exact sequence of cohomology from the exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_{\mathbb{CP}^1} \to \mathcal{O}^*_{\mathbb{CP}^1} \to 0$$

and observing that $H^1(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1}) = 0$ and $H^2(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1}) \cong H^{0,2}_{\overline{\partial}}(\mathbb{CP}^1) = 0$. $\deg(L)$ is also the integer obtained by pairing cohomology and homology, $c_1(L)(\mathbb{CP}^1)$. Further, in the case of the normal bundle to a submanifold Z, this degree coincides with the self-intersection number of Z, denoted $Z \cap Z$. Since the transition functions for $[E]|_E$ are inverse those for the hyperplane bundle, we have $E \cap E = -1$. Any other curve on M which is not E can be considered a curve on $\mathbb{CP}^2$, and these always have self-intersection number $+1$.

To obtain a holomorphic vector field X, we first define a flow ϕ_t on $\mathbb{CP}^2$ and then lift it to M. Note that ϕ_t must fix p, so that when p is blown up to E, ϕ_t lifts to M where it fixes E.

On U, set

$$\phi_t([1 : x : y]) = [1 : e^t x : e^t y]$$

so $\phi_t(p) = p$. Extend this to $\mathbb{CP}^1_\infty := \{[0 : x : y]\}$ by taking a limit

$$\phi_t([0 : x : y]) = \lim_{\lambda \to \infty} \phi_t([1 : \lambda x : \lambda y])$$

Note that ϕ_t fixes $\mathbb{CP}^1_\infty$; we have $[1 : e^t \lambda x : e^t \lambda y] = [e^{-t}\lambda^{-1} : x : y]$ for $\lambda \neq 0$, so as λ becomes large, we see taking the limit gives $[0 : x : y]$ again.

ϕ_t lifts to M by $\phi_t([1 : x : y], [\xi : \eta]) \mapsto ([1 : e^t x : e^t y], [\xi : \eta])$ and is defined on $\mathbb{CP}^1_\infty$ as above. The fixed points of ϕ_t on $\tilde{U}$ are precisely E, and everything outside $\tilde{U}$, namely $\mathbb{CP}^1_\infty$, is fixed. So if X is the vector field induced by ϕ_t

$$Fix(\phi_t) = E \cup \mathbb{CP}^1_\infty$$
$$\implies zero(X) = E \sqcup \mathbb{CP}^1_\infty$$

Both $E, \mathbb{CP}^1_\infty$ are isomorphic to $\mathbb{CP}^1$ so have genus 0 and Euler characteristic 2. As noted earlier $E \cap E = -1$, $\mathbb{CP}^1_\infty \cap \mathbb{CP}^1_\infty = +1$. It remins to compute $L_\lambda(X)$.

In coordinates about E, e.g. in V_0, the flow sends

$$(x_1^0, x_2^0) = (x, \eta/\xi) \mapsto (e^t x, \eta/\xi)$$

Since $\frac{d}{dt}(e^t x, \eta/\xi) = (e^t x, 0)$, X is locally given by $X = x_1^0 \frac{\partial}{\partial x_1^0}$. Since $E \cap V_0 = \{x_1^0 = 0\}$, we have a local frame for $TE|_{V_0}$ given by $\frac{\partial}{\partial x_2^0}$ therefore we can choose $\frac{\partial}{\partial x_1^0}$ as a local frame for $N_{M|E}$. So with respect to the basis $\frac{\partial}{\partial x_1^0}$ for $N_{M|E}$,

$$\nabla_{\frac{\partial}{\partial x_1^0}} X = 1 \cdot \frac{\partial}{\partial x_1^0} + x_1^0 \nabla_{\frac{\partial}{\partial x_1^0}} \frac{\partial}{\partial x_1^0}$$

On $E, x_1^0 = 0$ so we get $L_\lambda(X) = 1$ on E.

Next we find local coordinates about $\mathbb{CP}^1_\infty$. $\mathbb{CP}^1_\infty$ is contained in $U_1 \cup U_2$ where $U_i := \{[s_0 : s_1 : s_2] | s_i \neq 0\}$. U_1 has coordinates $(u, v) = (s_0/s_1, s_2/s_1)$ on which ϕ_t is

$$(u, v) \mapsto (e^{-t}u, v)$$

so $X = -u\frac{\partial}{\partial u}$ locally. $\mathbb{CP}^1_\infty \cap U_1 = \{u = 0\}$ so we can choose $N_{M|\mathbb{CP}^1_\infty}$ to be generated by $\frac{\partial}{\partial u}$. As in the calculation above, $\nabla_{\frac{\partial}{\partial u}} X = -\frac{\partial}{\partial u}$ on $\mathbb{CP}^1_\infty$ and $L_\lambda(X) = -1$.

So putting everything together

$$\begin{aligned}
\frac{1}{\pi^2} f_M(\pi c_1(M), X) &= \frac{1}{3}\left([2c_1(M)(E) + 2 - 2g(E)] - [2c_1(M)(\mathbb{CP}^1_\infty) + 2 - 2g(\mathbb{CP}^1_\infty)]\right) \\
&= \frac{2}{3}\left([\chi(E) + E \cap E] - [\chi(\mathbb{CP}^1_\infty) + \mathbb{CP}^1_\infty \cap \mathbb{CP}^1_\infty]\right) \\
&= \frac{2}{3}(2 - 1 - 2 - 1) \\
&= -\frac{4}{3} \neq 0
\end{aligned}$$

therefore $\mathbb{CP}^2(p)$ never admits a Kähler-Einstein metric.

3.3 Example 2

Let M be the blow-up of $\mathbb{CP}^1 \times \mathbb{CP}^1$ at the point $([1:0],[1:0])$. That is, M is the blow up of the image of the Segre embedding $\phi : \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^3$ at $p := [1:0:0:0]$. Let $X = \phi(\mathbb{CP}^1 \times \mathbb{CP}^1)$ and $w_0, \ldots, w_3$ be coordinates on $\mathbb{CP}^3$. So X is the zero set of $w_0w_3 - w_1w_2$. Note that if we blow up $\mathbb{CP}^1 \times \mathbb{CP}^1$ in any point p, using the automorphism group $PSL(2,\mathbb{C}) \times PSL(2,\mathbb{C}) \curvearrowright \mathbb{CP}^1 \times \mathbb{CP}^1$ we may assume $p = [1:0],[1:0]$. On X, p lies in the open set $U := U_0 \cap X$ on X, where U_i is the open set in $\mathbb{CP}^3$ consisting of points with nonzero ith coordinate. We have local parametrization

$$\begin{aligned}
f : \mathbb{C}^2 &\to U \\
(x, y) &\mapsto [1 : x : y : xy]
\end{aligned}$$

The projection onto the first factor is $\pi_1 : B_0\mathbb{C}^2 \to \mathbb{C}^2$ as earlier. Thus

$$\begin{aligned}
M &= X\backslash\{p\} \cup_{f\pi_1} B_0\mathbb{C}^2 \\
&= X\backslash\{p\} \cup \tilde{U}
\end{aligned}$$

where $\tilde{U} = \{[1 : x : y : xy], [\xi : \eta] | x\eta = y\xi\}$. The exceptional divisor is $E = \{[1:0:0:0], [\xi : \eta]\}$. Take

$$\begin{aligned}
V_0 &= \{([1 : x : y : xy], [\xi : \eta]) \in \tilde{U} | \xi \neq 0\} \\
V_1 &= \{([1 : x : y : xy], [\xi : \eta]) \in \tilde{U} | \eta \neq 0\}
\end{aligned}$$

Then V_0 has coordinates $(x_1^0, x_2^0) = (x, \eta/\xi)$ and V_1 has coordinates $(x_1^1, x_2^1) = (\xi/\eta, y)$, where E is given by $\{x = 0\}$ and $\{y = 0\}$ on V_0 and V_1 respectively. So as in Example 1, $E \cap E = -1$.

Define a flow on $\mathbb{CP}^3$ given by

$$\phi_t([w_0 : w_1 : w_2 : w_3]) = [w_0 : e^t w_1 : e^t w_2 : e^{2t} w_3]$$

This restricts to a flow on X and fixes p, so lifts to a flow on M fixing E. Also

$$\begin{aligned}[0 : e^t w_1 : e^t w_2 : e^{2t} w_3] &= [0 : w_1 : w_2 : e^t w_3] \\ &= [0 : w_1 : w_2 : w_3] \ \forall t \\ &\implies w_3 = 0\end{aligned}$$

Thus

$$\begin{aligned}&\{[0 : w_1 : w_2 : 0]\} \cap X = [0 : 1 : 0 : 0] \sqcup [0 : 0 : 1 : 0] \\ &\implies Fix(\phi_t) = E \sqcup q_1 \sqcup q_2\end{aligned}$$

where q_1, q_2 are the two fixed points. The flow locally on V_0 is

$$(x_1^0, x_2^0) \mapsto (e^t x_1^0, x_2^0)$$

thus $X := \frac{d}{dt}\phi_t = x_1^0 \frac{\partial}{\partial x_1^0}$ on V_0 gives

$$\begin{aligned}\nabla X|_E &= \nabla \left(x_1^0 \frac{\partial}{\partial x_1^0} \right) \Big|_E \\ &= dx_1^0 \frac{\partial}{\partial x_1^0} \\ \implies L_\lambda(X) &= 1 \text{ on } E\end{aligned}$$

where we used that $x_1^0 = 0$ on E so $\frac{\partial}{\partial x_1^0}$ is a generator for $N_{M|E}$. About q_1 we have open set U_1 and coordinates $(u, v) = (w_0/w_1, w_3/w_1)$ and then w_2 is determined by $w_0 w_3 = w_1 w_2$. Here

$$\begin{aligned}&\phi_t(u, v) = (e^{-t} u, e^t v) \\ &\implies X|_{U_1} = -u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}\end{aligned}$$

Then ∇X at q_1 is $\nabla X = \nabla \left(-u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}\right) = -du\frac{\partial}{\partial u} + dv\frac{\partial}{\partial v}$ since X vanishes at q_1. As q_1 is a point, its normal bundle is all of $T_{q_1}M$ so $L_\lambda(X)$ is a 2×2 matrix given by $L_\lambda(X) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ therefore $tr(L_\lambda(X)) = 0$.

The calculation for q_2 is identical, with coordinates $(u, v) = (w_0/w_2, w_3/w_2)$, $\phi_t(u, v) = (e^{-t}u, e^t v)$, $X|_{U_2}$ and $L_\lambda(X)$ are the same, so $tr(L_\lambda(X)) = 0$. So the points do not contribute to f_M. Then

$$\begin{aligned}f_M(\pi c_1(M), X) &= \frac{\pi^2}{3}(2c_1(M)(E) + 2 - 2g(E)) \\ &= \frac{2\pi^2}{3}(\chi(E) + E \cap E + 1) \\ &= \frac{2\pi^2}{3}(2 - 1 + 1) = \frac{4\pi^2}{3} \neq 0\end{aligned}$$

So $\mathbb{CP}^1 \times \mathbb{CP}^1(p)$ never admits a Kähler-Einstein metric.

3.4 Fano surfaces

We've shown above that $\mathbb{CP}^1 \times \mathbb{CP}^1(p)$ and $\mathbb{CP}^2(p)$ do not admit Kähler-Einstein metrics. $\mathbb{CP}^n$ admits the Fubini-Study metric, given by (where $||z||^2 = \sum_{i=1}^{n} |z_i|^2$ for local coordinates $z_1, \ldots, z_n$ on $\mathbb{CP}^n$)

$$\omega_{FS} = \frac{i}{2}\partial\overline{\partial}\log(1+||z||^2) = \frac{i}{2}\left(\frac{\sum_i dz_i \wedge d\overline{z}_i}{1+||z||^2} - \frac{\sum_i \overline{z}_i dz_i \wedge \sum_j z_j d\overline{z}_j}{(1+||z||^2)^2}\right) \tag{18}$$

$$\omega_{FS}^n = n!\left(\frac{i}{2}\right)^n \frac{dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge dz_n \wedge d\overline{z}_n}{(1+||z||^2)^{n+1}} \tag{19}$$

The latter equality can be seen as follows:

$$\begin{aligned}
\left(\sum_i \overline{z}_i dz_i \wedge \sum_j z_j d\overline{z}_j\right)^2 &= \sum_{i,j,k,l} \overline{z}_i z_j \overline{z}_k z_l dz_i \wedge d\overline{z}_j \wedge dz_k \wedge d\overline{z}_l \\
&= \sum_{i<k} \overline{z}_i z_j \overline{z}_k z_l dz_i \wedge d\overline{z}_j \wedge dz_k \wedge d\overline{z}_l + \sum_{i>k} \overline{z}_i z_j \overline{z}_k z_l dz_i \wedge d\overline{z}_j \wedge dz_k \wedge d\overline{z}_l \\
&= \sum_{i<k} \overline{z}_i z_j \overline{z}_k z_l (dz_i \wedge d\overline{z}_j \wedge dz_k \wedge d\overline{z}_l - dz_i \wedge d\overline{z}_j \wedge dz_k \wedge d\overline{z}_l) = 0
\end{aligned}$$

Thus in ω_{FS}^n, the terms $\left(\frac{\sum_i \overline{z}_i dz_i \wedge \sum_j z_j d\overline{z}_j}{(1+||z||^2)^2}\right)^k$ for $k \geq 2$ are zero. Then

$$\begin{aligned}
(-2i\omega_{FS})^n &= \left(\frac{\sum_i dz_i \wedge d\overline{z}_i}{1+||z||^2}\right)^n - n\left(\frac{\sum_i dz_i \wedge d\overline{z}_i}{1+||z||^2}\right)^{n-1} \wedge \frac{\sum_i \overline{z}_i dz_i \wedge \sum_j z_j d\overline{z}_j}{(1+||z||^2)^2} \\
&= \frac{\left(\sum_i dz_i \wedge d\overline{z}_i\right)^n (1+||z||^2)}{(1+||z||^2)^{n+1}} - ||z||^2 \frac{\left(\sum_i dz_i \wedge d\overline{z}_i\right)^n}{(1+||z||^2)^{n+1}} \\
&= \frac{\left(\sum_i dz_i \wedge d\overline{z}_i\right)^n}{(1+||z||^2)^{n+1}}
\end{aligned}$$

Thus

$$\begin{aligned}
Ric(\omega_{FS}) &= -\frac{i}{2}\partial\overline{\partial}\log(\omega_{FS}^n) \\
&= -\frac{i}{2}\partial\overline{\partial}\log\left(\frac{1}{(1+||z||^2)^{n+1}}\right) \\
&= (n+1)\frac{i}{2}\partial\overline{\partial}\log(1+||z||^2) \\
&= (n+1)\omega_{FS}
\end{aligned}$$

So $\mathbb{CP}^n$ is Kähler-Einstein.

$M := \mathbb{CP}^1 \times \mathbb{CP}^1$ is also Kähler-Einstein. Let $\pi_i : M \to \mathbb{CP}^1$ be the projection onto the ith factor. M admits a Kähler structure as a product of two Kähler manifolds, induced by the metric $g := g_{FS}^1 + g_{FS}^2$ where

$$(g_{FS}^1 + g_{FS}^2)(u,v) = g_{FS}^1(\pi_{1*}u, \pi_{1*}v) + g_{FS}^2(\pi_{2*}u, \pi_{2*}v)$$

So in local coordinates (z_1, z_2) on $\mathbb{CP}^1 \times \mathbb{CP}^1$

$$\omega = \frac{i}{2}\left(\frac{dz_1 \wedge d\overline{z}_1}{(1+|z_1|^2)^2} + \frac{dz_2 \wedge d\overline{z}_2}{(1+|z_2|^2)^2}\right) = \omega_{FS}^1 + \omega_{FS}^2$$

Then

$$
\begin{aligned}
Ric(\omega) &= -\frac{i}{2}\partial\overline{\partial}\log\omega^2 \\
&= -\frac{i}{2}\partial\overline{\partial}\log(\omega_{FS}^1 \wedge \omega_{FS}^2) \\
&= -\frac{i}{2}\partial\overline{\partial}\left[\log\left(\frac{1}{(1+|z_1|^2)^2}\right) + \log\left(\frac{1}{(1+|z_2|^2)^2}\right)\right] \\
&= 2(\omega_{FS}^1 + \omega_{FS}^2) = 2\omega
\end{aligned}
$$

Theorem 19 ([Hit75]). *The Fano surfaces are $\mathbb{CP}^1 \times \mathbb{CP}^1$ and $\mathbb{CP}^2(p_1, \ldots, p_k)$ blown up at k points in general position for $0 \leq k \leq 8$.*

Note that $\mathbb{CP}^2(p_1, p_2) \cong \mathbb{CP}^1 \times \mathbb{CP}^1(p)$, see [GH94, pg 478–450].

Ideas in the proof of 19. Hitchin's proof showed that $c_1(M) > 0$ implies M is birational to $\mathbb{CP}^2$, and further blowing down M does not change the sign of $c_1(M)$ so we may look at the minimal models for rational surfaces, which are $\mathbb{CP}^2$ and $F_n := \mathbb{P}(H^n \oplus \mathbf{1})$, $n \neq 1$, where H is the hyperplane bundle over $\mathbb{CP}^1$. Using the Riemann-Roch theorem for surfaces for a non-singular rational curve D on M gives

$$-c_1(M) \cdot [D] + [D]^2 = -2$$

so using $c_1(M) > 0$ one can show this implies $[D]^2 > -2$. The manifold F_n has a rational curve with self-intersection $-n$ so we can exclude all F_n except $F_0 \cong \mathbb{CP}^1 \times \mathbb{CP}^1$. As $\mathbb{CP}^1 \times \mathbb{CP}^1(p) \cong \mathbb{CP}^2(p_1, p_2)$, we can assume we blow up in points on $\mathbb{CP}^2$. The first Chern classes of a blow up $\pi : \hat{M} \to M$ of M in a point are related by

$$c_1(\hat{M}) = \pi^* c_1(M) - [E]$$

where E is the exceptional divisor. If we blow up $\mathbb{CP}^2$ in k points then this implies

$$c_1(\mathbb{CP}^2(p_1, \ldots, p_k))^2 = 9 - k > 0$$

using the result that $c_1(\mathbb{CP}^2)$ corresponds to the integer 3. So we require $k \leq 8$. We must blow up in distinct points since blowing up in a point gives an exceptional divisor of self-intersection -1 and blowing up again in the same point gives a curve of self-intersection -2, but $[C]^2 > -2$ for all non-singular rational curves C.

The following theorem and the previous examples answer the Kähler-Einstein question for Fano surfaces.

Theorem 20 ([Tia00, pg 87]). *The Fano surfaces $\mathbb{CP}^2(p_1, \ldots, p_k)$ for $3 \leq k \leq 8$ and p_i in general position all admit Kähler-Einstein metrics.*

4 Asymptotic Chow Stability

The existence of KE metrics is also related to stability. We can generalize to the case of a compact complex manifold M with positive line bundle L. Then the pair (M, L) is called a polarized manifold. $c_1(L)$ is represented by a positive closed $(1,1)$ form, locally given by $\frac{i}{2}\sum_{i,j} g_{i\bar{j}} dz_i \wedge d\bar{z}_j$, where $g_{i\bar{j}}$ is a positive definite hermitian matrix. Then $g := \sum_{i,j} g_{i\bar{j}} dz_i \otimes d\bar{z}_j$ is a hermitian Kähler metric and M is a Kähler manifold. Note that since L is positive, for sufficiently large k we can embed $\iota_k : M \to \mathbb{CP}^{N_k}$ via sections of L^k. This is the statement of the Kodaira Embedding theorem, [GH94, pg 181].

We seek to find a constant scalar curvature Kähler (cscK) metric in $c_1(L)$, which is a generalization of finding a KE metric in $c_1(M)$. I will describe some results related to this question and then give the background behind them.

Let $Aut(M, L)$ denote the subgroup of $Aut(L)$ consisting of automorphisms of L which commute with the $\mathbb{C}^*$-action on fibers. In particular, these descend to automorphisms of M, so $Aut(M, L)$ can be identified with a subgroup of $Aut(M)$. When $Aut(M, L)$ is discrete its Lie algebra is trivial. Donaldson proved in [Don01], for the sequence of metrics $\omega_k := \frac{2\pi}{k}\iota_k^*(\omega_{FS})$,

Theorem 21 ([Don01]). *Suppose (M, L) is a polarized manifold and $Aut(M, L)$ is discrete. If ω is a cscK metric in $2\pi c_1(L)$, then (M, L^k) is balanced for all sufficiently large k and the sequence of metrics ω_k converges in C^∞ to ω, as $k \to \infty$.*

Corollary 22. *When $Aut(M, L)$ is discrete, if $2\pi c_1(L)$ admits a cscK metric it is unique.*

Wang showed using moment maps and symplectic reduction

Theorem 23 ([Wan04]). *Let (M, L) be polarized by a very ample line bundle L, with embedding $M \to \mathbb{P}^N$ via L. Then (M, L) is Chow polystable if and only if it can be balanced.*

Theorem 23 was originally due to Zhang, and there is a proof by Paul as well.

Corollary 24. *Asymptotic Chow stability of a polarized manifold (M, L) is an obstruction to the existence of cscK metrics in $c_1(L)$ when $Aut(M, L)$ is discrete.*

4.1 Background

4.1.1 Chow form and asymptotic Chow stability

The Chow form gives a way of parametrizing polarized manifolds. Chow polystability is defined in terms of stability of the Chow form. There are two equivalent ways of defining the Chow form. Let (M, L) be a polarized manifold as above with embedding $\iota_k : M \to \mathbb{CP}^{N_k}$. Set $n := \dim_{\mathbb{C}} M$ and d_k is the degree of $\iota_k(M) \subset \mathbb{CP}^{N_k}$.

1) Consider the set of all $N_k - (n+1)$ dimensional subspaces V of $\mathbb{CP}^{N_k}$, i.e. points in $\mathbb{G}(N_k - n, N_k + 1)$, the Grassmannian of $N_k - n$ dimensional subspaces of $\mathbb{C}^{N_k+1}$. Given a set of coordinates $e_1, \ldots, e_n$ on $\mathbb{C}^n$, we have a set of coordinates $e_{i_1} \wedge \ldots \wedge e_{i_r}$, $1 \leq i_1 < \ldots < i_r \leq n$ on $\bigwedge^r \mathbb{C}^n$, called the Plücker coordinates. Then $\mathbb{G}(N_k - n, N_k + 1)$ embeds into $\mathbb{P}(\bigwedge^{N_k - n} \mathbb{C}^n)$ by sending a space V spanned by $e_1, \ldots, e_{N_k - n}$ to the one dimensional space spanned by $e_1 \wedge \ldots \wedge e_{N_k - n}$ in $\bigwedge^{N_k - n} \mathbb{C}^n$. This is called the Plücker embedding.

V does not generically intersect M since $\dim V + \dim M = \dim \mathbb{CP}^{N_k} - 1$. However, the set of V such that $V \cap M \neq \phi$ forms an irreducible codimension one subvariety of $\mathbb{G}(N_k - n, N_k + 1)$

and hence is defined by the vanishing of some polynomial degree d_k in the Plücker coordinates of $\mathbb{G}(N_k - n, N_k + 1)$. This is proved in [GKZ94, Chapter 3, §2, A and B]. This polynomial is the Chow point or Chow form. It is a point in the Chow space

$$\mathcal{CHOW}_{\mathbb{P}^{N_k}}(n, d_k) \subset \mathbb{P}H^0(\mathbb{G}(N_k - n, N_k + 1), \mathcal{O}_{\mathbb{G}}(d_k))$$

whose points parametrize polarized varieties (M, L) where L^k induces an embedding $\iota_k : M \to \mathbb{P}^{N_k}$, $n = \dim_{\mathbb{C}} M$ and $d_k = \deg(\iota_k(M))$. This is the approach taken in [Wan04].

2) Following Futaki [Fut11], let $V_k = H^0(M, L^k)^*$ so $N_k + 1 = \dim V_k$. Elements in $\mathbb{P}(V_k^*)$ define hyperplanes in $\mathbb{P}(V_k)$ so if

$$D_M := \{(H_1, \ldots, H_{n+1}) \in \mathbb{P}(V_k^*) \times \ldots \times \mathbb{P}(V_k^*) | H_1 \cap \ldots \cap H_{n+1} \cap M \neq \phi\}$$

then D_M is a divisor in $\mathbb{P}(V_k^*) \times \ldots \times \mathbb{P}(V_k^*)$ defined by a polynomial in $(Sym^{d_k}(V_k))^{\otimes n+1}$, also called the Chow form. These two definitions of the Chow form are equivalent as described by Mumford [Mum77, pg 16 – 17].

Let $Chow(M, L^k)$ denote the kth Chow form of (M, L) as defined above. Note that $G := SL(N_k + 1)$ acts on $\mathbb{CP}^{N_k}$, which corresponds to changing the chosen basis for $H^0(M, L^k)$. This induces an action on $(Sym^{d_k}(V_k))^{\otimes n+1}$ or $\mathbb{P}H^0(\mathbb{G}(N_k - n, N_k + 1), \mathcal{O}_{\mathbb{G}}(d_k))$ so we can consider the orbit of $Chow(M, L^k)$. Chow stability of M corresponds with the geometric invariant theory (GIT) stability of $Chow(M, L^k)$ with respect to this G-action.

Definition 25. Let (M, L) be a polarized manifold.

1. M is Chow polystable w.r.t. L^k if the orbit of $Chow(M, L^k)$ under G is closed.
2. M is Chow stable w.r.t L^k if it is polystable and the stabilizer of $Chow(M, L^k)$ is finite.
3. M is asymptotically Chow stable w.r.t. L if there exists a $k_0 > 0$ such that for all $k \geq k_0$, M is Chow stable w.r.t L^k.

4.1.2 Moment map and symplectic quotient

I learned the following background from [DK90], [Tho06], [Wan04] and Wikipedia.

Definition 26. A symplectic manifold M is a smooth manifold equipped with a closed, non-degenerate global 2-form ω.

Definition 27. We say an automorphism g of M is symplectic or a symplectomorphism if it preserves ω, i.e. $g^*\omega = \omega$.

For example a Kähler manifold is symplectic, where ω is its Kähler metric. If $M = \mathbb{CP}^N$ is equipped with the Fubini-Study metric $\omega_{FS} = \frac{i}{2}\partial\overline{\partial}\log(||z||^2)$ then $g \in GL(N+1, \mathbb{C})$ preserves ω_{FS} if and only if $g \in U(N+1)$ since

$$||g(z)||^2 = \overline{z^T g^T} gz = \overline{z}^T z^T = ||z||^2$$

for all $z = (z_1, \ldots, z_{N+1})$ if and only if $\overline{g}^T g = I$ i.e. $g \in U(N+1)$.

Let G be a compact Lie group acting on M by symplectomorphisms. Let $\mathfrak{g}$ denote the Lie algebra of G. Each element $\xi \in \mathfrak{g}$ defines a vector field $\sigma(\xi)$ on M by considering its infinitesimal action. On T_zM the vector $\sigma_z(\xi)$ is defined to be

$$\sigma_z(\xi) := \frac{d}{dt}\bigg|_{t=0} \exp(t\xi) \cdot z$$

where $\exp(t\xi)$ is the 1-parameter subgroup (1-PS) of G induced by ξ via the exponential map. Since G acts by symplectomorphisms the Lie derivative $L_{\sigma(\xi)}\omega = 0$. Thus using Cartan's formula and the fact that ω is closed we obtain

$$\begin{aligned} 0 = L_{\sigma(\xi)}\omega &= d(\iota_{\sigma(\xi)}\omega) + \iota_{\sigma(\xi)} d\omega \\ &= d(\iota_{\sigma(\xi)}\omega) \end{aligned}$$

So if $H^1(M, \mathbb{R}) = 0$, then $\iota_{\sigma(\xi)}\omega$ is exact. For example, when M is Kähler and $c_1(M) > 0$, the Calabi-Yau theorem states that M admits a metric with positive Ricci curvature, which implies M is simply connected ([Tia00, Remark 2.14]). So $H^1(M, \mathbb{R}) = 0$ in that case.

Then we may write $\iota_{\sigma(\xi)}\omega = dm_\xi$ for some function m_ξ on M. Note that in general if $df = 0$, then f lies in the kernel of d, which consists of locally constant functions on M, so for M simply connected f is constant. That is, m_ξ is unique up to a constant.

Definition 28. A moment map for the action of G on M is a map

$$m : M \to \mathfrak{g}^*$$

where $\langle m(z), \xi \rangle = m_\xi(z)$, such that

$$d \langle m, \xi \rangle = \iota_{\sigma(\xi)}\omega \tag{20}$$

The final equation is an expression in terms of $z \in M$, where $\langle m, \xi \rangle (z) = \langle m(z), \xi \rangle$, and $(\iota_{\sigma(\xi)}\omega)(z) = \iota_{\sigma_z(\xi)}\omega$. That is, a moment map for the action of G on M is obtained by combining all the components m_ξ into a map. Here $\langle , \rangle$ denotes the evaluation pairing $\mathfrak{g} \times \mathfrak{g}^* \to \mathbb{R}$. If we have a bi-invariant non-degenerate pairing on $\mathfrak{g} \times \mathfrak{g}$, such as taking the trace of the product of two matrices in the Lie algebra of the special linear group, then we can identify $\mathfrak{g}$ with its dual via this pairing and consider m as a map $M \to \mathfrak{g}$.

Definition 29. G acts on itself by conjugation, which induces the adjoint action on $\mathfrak{g}$.

$$\psi(g) : G \to G$$

$$h \mapsto ghg^{-1}$$

This induces $\mathrm{Ad}(g) := (d\psi(g))_e : T_eG \to T_eG$, the adjoint action of G on $\mathfrak{g}$. Then m is G-equivariant if

$$\langle m(g \cdot z), \xi \rangle = \langle \mathrm{Ad}(g)^* m(z), \xi \rangle, \qquad \forall z \in M, \forall \xi \in \mathfrak{g}$$

where the right hand side is $\left\langle m(z), \mathrm{Ad}(g^{-1}) \cdot \xi \right\rangle$.

Recall the m_ξ are unique up to constants. It is possible to choose these constants such that m is a G-equivariant moment map. Then m is uniquely defined up to addition of a central element in $\mathfrak{g}^*$, [DK90],[Tho06].

The moment map generalizes the notion of angular and linear momentum, hence its name. The possible locations of a particle in 3-space form its configuration space, $\mathbb{R}^3$. The phase space corresponding to this configuration space is $M := T^*\mathbb{R}^3$; each point of M corresponds to a unique state of the particle, describing its position and momentum. The group G of translations $\mathbb{R}^3$ and rotations $SO(3)$ acts on M and the components of the moment map $m : M \to \mathfrak{g}^*$ are the components of angular and linear momentum in each of the three directions.

Assuming m is G-equivariant, G acts on $m^{-1}(0)$.

Definition 30. The symplectic quotient of M by G is

$$M//G := m^{-1}(0)/G$$

When M is a compact Kähler manifold we consider the action of a compact Lie group G and extend the action to its complexification $G^{\mathbb{C}}$, where $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} + i\mathfrak{g}$, via the almost complex structure J. So $\sigma(i\xi) = J\sigma(\xi)$. For example, $SU(N+1) \subset SL(N+1,\mathbb{C})$, $\mathfrak{sl}(N+1,\mathbb{C}) =$ trace 0 matrices over $\mathbb{C}$ and $\mathfrak{su}(N+1) =$ trace 0 skew-hermitian matrices, as a Lie algebra over $\mathbb{R}$. Since any trace 0 matrix A can be written as a sum of trace 0 hermitian and skew-hermitian matrices $A = \frac{1}{2}(A+\overline{A}^T)+\frac{1}{2}(A-\overline{A}^T)$, we have $\mathfrak{sl}(N+1,\mathbb{C}) = \mathfrak{su}(N+1)+i\cdot\mathfrak{su}(N+1)$.

As an example $(\mathbb{P}^N, \omega_{FS})$ has a canonical moment map μ_{FS}

$$\mu_{FS} : (z_0 : \ldots : z_N) \mapsto \frac{1}{2i}\left(\frac{z_i\overline{z}_j}{||z||^2} - \frac{\delta_{ij}}{N+1}\right) \in \mathfrak{su}(N+1)$$

Note that the trace of the image is $\frac{1}{2i}\left(\frac{||z||^2}{||z||^2} - \frac{\delta_{ii}}{N+1}\right) = 1-1 = 0$ as required.

Proof follows [MFK94, Example 8.1(ii)] and [Kir84, Lemma 2.5]. As noted above $U(N+1)$ acts on $\mathbb{P}^N$ by symplectomorphisms. Note that $\frac{1}{2i}\left(\frac{||z||^2}{||z||^2} - \frac{\delta_{ii}}{N+1}\right)$ can be identified with its dual element in $\mathfrak{su}(N+1)^*$ via the Killing form $-tr(a\cdot b)$ for $a, b \in \mathfrak{su}(N+1)$. This is symmetric and bilinear. So

$$\begin{aligned}\langle \mu_{FS}(z), a\rangle_{\mathfrak{su}} &= -\frac{i}{2}tr\left(\left(\frac{z_i\overline{z}_j}{||z||^2} - \frac{\delta_{ij}}{N+1}\right)\cdot a_{jk}\right) \\ &= \frac{1}{2i}\left(\frac{z_i\overline{z}_j\cdot a_{ji}}{||z||^2} - \frac{a_{ii}}{N+1}\right)\end{aligned}$$

A G-equivariant moment map is unique up to addition of an element which is central in the Lie algebra, and in this case the central elements of $\mathfrak{u}(N+1)$ are constant scalar multiples of the identity which are skew-hermitian, i.e. elements of the form $i\cdot rI_{N+1}$ for $r \in \mathbb{R}$. Thus we've chosen to add the constant $\frac{i}{2}\frac{\delta_{ij}}{N+1}$ above so the image of the moment map is in $\mathfrak{su}(N+1)$, i.e. it is trace free.

First note that μ_{FS} is independent of the non-zero representative chosen for z in $\mathbb{C}^{N+1}$; this is because if we chose λz instead then the λ^2 cancel in the numerator and denominator. μ_{FS} is $SU(N+1)$-equivariant since for $g \in SU(N+1)$ and $z \in \mathbb{P}^N$,

$$\begin{aligned}\langle \mu_{FS}(g\cdot z), a\rangle &= \frac{1}{2i}\left(\frac{\overline{z}^T\overline{g}^T agz}{||z||^2} - \frac{tr(a)}{N+1}\right) \\ &= \frac{1}{2i}\left(\frac{\overline{z}^T g^{-1}agz}{||z||^2} - \frac{tr(g^{-1}ag)}{N+1}\right) \\ &= \langle \mathrm{Ad}(g)^*\mu_{FS}(z), a\rangle\end{aligned}$$

Since $U(N+1)$ is transitive on $\mathbb{P}^N$, it suffices to prove that Equation 20 holds at the point $p = (1 : 0 : \ldots : 0)$ which corresponds to the origin in the coordinate chart on U_0, the set of points with nonzero first coordinate. The vector field induced by $a \in \mathfrak{su}(N+1)$ takes the value at p in coordinates $z_1, \ldots, z_n$ on U_0 given by

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} e^{ta} \cdot p &= \left.\frac{d}{dt}\right|_{t=0} (I + ta + O(t^2)) \cdot p \\ &= a \cdot p \\ &= (a_{10}, \ldots, a_{n0})\end{aligned}$$

From Equation 18, at p

$$\omega_{FS} = \frac{i}{2} \sum_{i=1}^{n} dz_i \wedge d\overline{z}_i$$

Thus in coordinates on U_0 (n.b. $||p||^2 = 1$)

$$\begin{aligned}d_p \langle \mu_{FS}, a \rangle &= d_p \left[\frac{1}{2i} \left(\frac{z_i \overline{z}_j \cdot a_{ji}}{||z||^2} - \frac{a_{ii}}{N+1} \right) \right] \\ &= \frac{1}{2i} d_p \left(z_i \overline{z}_j \cdot a_{ji} \right) \\ &= \frac{1}{2i} \sum_{i,j=0}^{n} \sum_{k=1}^{n} a_{ji} [\delta_{ik} \overline{z}_j dz_k + \delta_{jk} z_i d\overline{z}_k]_p \\ &= \frac{1}{2i} \sum_k a_{0k} dz_k + a_{k0} d\overline{z}_k \\ &= \frac{i}{2} \sum_k \overline{a_{k0}} dz_k - a_{k0} d\overline{z}_k \\ &= \iota_{\sigma(a)} \omega_{FS}|_p\end{aligned}$$

using that a is skew-hermitian. □

4.2 Chow polystability is equivalent to balanced

The space of Chow points admits the structure of a Kähler manifold, with Kähler form Ω as follows. Each $f \in \mathcal{CHOW}_{\mathbb{P}^{N_k}}(n, d_k)$ is a symmetric degree d_k polynomial in the Plücker coordinates of $\mathbb{G}(N_k - n, N_k + 1)$, and parameterizes a polarized variety (M, L). The tangent space $T_f \mathcal{CHOW}_{\mathbb{P}^{N_k}}(n, d_k)$ can be identified with $\Gamma(M, T\mathbb{P}^{N_k}|_M)$. In order to see how $\mathcal{CHOW}_{\mathbb{P}^{N_k}}(n, d_k)$ varies infinitesimally at f, we can look at the "velocity" of each point on M, since M corresponds to the point f. That is, a tangent vector at f in the Chow space corresponds to assigning a "direction" everywhere on M, i.e. a global section of $T\mathbb{P}^{N_k}|_M$, because we know $M \hookrightarrow \mathbb{P}^{N_k}$ ([GKZ94, §4.3]).

Define Ω_f on $u, v \in \Gamma(M, T\mathbb{P}^{N_k}|_M)$ by

$$\Omega_f(u, v) = \int_M \frac{\iota_v(\iota_u(\omega_{FS}^{n+1}))}{(n+1)!} \tag{21}$$

where ι_u denotes contraction with u.

Lemma 31 ([Wan04, Proposition 17]). *The map*

$$\mu_\Omega : \mathcal{CHOW}_{\mathbb{P}^{N_k}}(n, d_k) \to \mathfrak{su}(N_k+1)$$

given by

$$Chow(M, L^k) \mapsto \int_M \mu_{FS} \frac{\omega_{FS}^n}{n!}$$

is a moment map for $(\mathcal{CHOW}_{\mathbb{P}^{N_k}}(n, d_k), \Omega)$.

We can check that

$$d\langle \mu_\Omega, \xi\rangle = \iota_{\sigma(\xi)}\Omega, \qquad \forall \xi \in \mathfrak{su}(N_k+1) \tag{22}$$

holds pointwise at each Chow point f. The right side is computed by evaluating at some $Y \in T_f\mathcal{CHOW}_{\mathbb{P}^{N_k}}(n, d_k)$, which can be identified as an element in $\Gamma(M, T\mathbb{P}^{N_k}|_M)$. The left side of (22) is computed by taking a path f_t in the Chow space with $f_0 = f$ such that its "velocity" at $t = 0$ is Y (cf [Wan04]). So μ_Ω is the required moment map. In particular, μ_Ω inherits G-equivariance from μ_{FS} so we can form the symplectic quotient

$$\mathcal{CHOW}_{\mathbb{P}^{N_k}}(n, d_k)//SU(N+1) = \mu_\Omega^{-1}(0)/SU(N+1)$$

Definition 32. We say that (M, L^k) <u>can be balanced</u> if there exists a choice of basis for embedding M in $\mathbb{CP}^{N_k}$ such that $Chow(M, L^k)$ is a zero of the moment map μ_Ω.

Sketch proof of Theorem 23. There are several intermediate results in the following which I state without proof, hence I've labelled this as a sketch proof.

Since L is very ample we can assume $k = 1$ and drop the k's. By the Hilbert-Mumford Criterion [MFK94, §2.1], to check polystability of $Chow(M, L)$ with respect to the $SU(N+1)$-action we need only check it for all 1-parameter subgroups outside its stabilizer.

In order to define an action of a one-parameter subgroup, we need something which is invariant. Define $M_\infty := \lim_{t\to\infty} e^{it\xi} \cdot M$ for $\xi \in \mathfrak{su}(N+1) - \mathfrak{aut}(M, L)$. Here $\mathfrak{aut}(M, L)$ is the Lie algebra of $Aut(M, L)$. Then M_∞ is invariant under the action of the 1-PS $\{e^{t\xi}\}_{t\in\mathbb{C}}$, since $SU(N+1)$ acts by symplectomorphisms and we've made M_∞ invariant under $i\mathfrak{su}(N+1)$ as well.

Definition 33. The <u>λ-weight</u> $\rho(x)$ of a $\mathbb{C}^*$-action on an element x in an invariant one-dimensional space is the exponent of the eigenvalue, where $\lambda \in \mathbb{C}^*$ acts by

$$\lambda \cdot x = \lambda^{\rho(x)} x$$

In general, given a $\mathbb{C}^*$-action on a space V, there are eigenvectors $v_1, \dots, v_m$ with eigenspaces of dimension d_i with respect to the action such that $\lambda \in \mathbb{C}^*$ acts by $\lambda \cdot v_i = \lambda^{a_i} \cdot v_i$. Then the <u>$\lambda$-weight</u> of the action on this space is $\sum_i a_i d_i$. Equivalently, $\mathbb{C}^* \curvearrowright V \implies \mathbb{C}^* \curvearrowright \bigwedge^{top} V$ and we can compute the weight on $\bigwedge^{top} V$ as in the one-dimensional case.

Definition 34. The <u>ξ-weight</u> of $Chow(M, L)$ is defined to be the weight of the action induced by ξ on the one dimensional space $Z := \mathcal{O}_{\mathcal{CHOW}_{\mathbb{P}^N}(n,d)}(1)$ at the point $Chow(M_\infty, L)$.

Chow polystability was defined earlier by looking at orbits of $Chow(M, L)$. There is an equivalent notion of polystability by [MFK94]: M is Chow polystable if the ξ-weight of $Chow(M, L)$ is negative for all $\xi \in \mathfrak{su}(N+1) - \mathfrak{aut}(M, L)$.

We use the following two results:

Proposition 35 ([Mum77, Prop 2.11]). *Suppose M is an n-dimensional manifold embedded in $\mathbb{P}^N$ via very ample line bundle L and is fixed by a 1-PS ξ of $SL(N+1)$. Let R_N be the degree N part of the projective homogenous coordinate ring for M. Let a_M be the ξ-weight of $Chow(M,L)$ and r_N^M the ξ-weight of R_N. Then for large N, r_N^M is represented by a polynomial in N of degree at most $n+1$, with normalized leading coefficient a_M, i.e. the leading coefficient is $a_M/n!$.*

In the notation of [Wan04], the leading coefficient of r_N^M is denoted $w_{M_\infty,0}(\xi)$.

Theorem 36 ([Wan04, Theorem 26]).

$$\frac{w_{M_\infty,0}(\xi)}{n+1} = \lim_{t\to\infty} \left\langle \mu_\Omega(e^{it\xi}\cdot Chow(M,L)), \xi \right\rangle_{\mathfrak{su}}$$

Thus the ξ-weight of $Chow(M,L)$ has the same sign as $\lim_{t\to\infty} \left\langle \mu_\Omega(e^{it\xi}\cdot Chow(M,L)), \xi \right\rangle_{\mathfrak{su}}$. Hence Chow polystability of M is equivalent to

$$-\lim_{t\to\infty} \left\langle \mu_\Omega(e^{it\xi}\cdot Chow(M,L)), \xi \right\rangle_{\mathfrak{su}} > 0$$

for all $\xi \in \mathfrak{su}(N+1) - \mathfrak{aut}(M,L)$.

We then use the following results from [DK90, Section 6.5.2], also described in [Tho06]. Let $G = SU(N+1)$. We can lift the G-action on $\mathcal{CHOW}_{\mathbb{P}^N}(n,d)$ to one on Z, which extends to $G^{\mathbb{C}}$. We have a projection map

$$\pi : Z \to \mathcal{CHOW}_{\mathbb{P}^N}(n,d)$$

$$\pi_* : TZ \to T\mathcal{CHOW}_{\mathbb{P}^N}(n,d)$$

Then TZ is the direct sum of the vertical subspace, which is $\ker \pi_*$, and a horizontal subspace orthogonal to the vertical subspace, with respect to the induced inner product on Z. Tangent vectors on the Chow space have a horizontal lift to the horizontal subspace. Let $\widetilde{\sigma(\xi)}$ be the lift to this horizontal subspace, which projects to $\sigma(\xi)$ via π_*.

Recall $\langle \mu_\Omega, \xi\rangle$ is the function corresponding to the vector field $\sigma(\xi)$ on $\mathcal{CHOW}_{\mathbb{P}^N}(n,d)$, i.e. $d\langle \mu_\Omega, \xi\rangle = \iota_{\sigma(\xi)}\Omega$. Then the infinitesimal action of ξ on Z is defined at a point γ over z (where z corresponds to the point $Chow(M,L)$)

$$\widetilde{\sigma_z(\xi)} + i\langle \mu_\Omega(z), \xi\rangle \gamma \tag{23}$$

Z carries a G-invariant hermitian metric induced by Ω so $||g\cdot\gamma|| = ||\gamma||$ for all $g \in G$. So when considering how $||g\cdot\gamma||$ changes along a 1-PS, we need only look at $i\xi \in i\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$.

Let

$$H_\xi(t) = \log ||e^{it\xi}\cdot\gamma||^2$$

defined by exponentiating the infinitesimal action above. Then (defining $z_t := e^{it\xi}z$)

$$\begin{aligned} H'_\xi(t) &= \frac{2\left\langle \frac{d}{dt}(e^{it\xi}\cdot\gamma), e^{it\xi}\gamma\right\rangle}{||e^{it\xi}\cdot\gamma||^2} \\ &= \frac{2\left\langle i\langle \mu_\Omega(z_t), i\xi\rangle e^{it\xi}\gamma, e^{it\xi}\gamma\right\rangle}{||e^{it\xi}\cdot\gamma||^2} \\ &= -2\langle \mu_\Omega(z_t), \xi\rangle \end{aligned}$$

using Equation 23 and orthogonality of horizontal and vertical subspaces. If γ is a point in some $G^{\mathbb{C}}$-orbit $G^{\mathbb{C}} \cdot \gamma_0$, then it is a critical point of $\log ||g \cdot \gamma_0||^2$ if and only if $\mu_\Omega(\pi(\gamma)) = 0$. Critical points of $H_\xi(t)$ are minima; note that

$$\begin{aligned} H''_\xi(t) &= -2d \langle \mu_\Omega, \xi \rangle |_{z_t}(\sigma_{z_t}(i\xi)) \\ &= -2(\iota_{\sigma_{z_t}(\xi)}\Omega)(J\sigma_{z_t}(\xi)) \\ &= -2\Omega(\sigma_{z_t}(\xi), J\sigma_{z_t}(\xi)) \\ &= 2||\sigma_{z_t}(\xi)||^2 \end{aligned}$$

Thus $H_\xi(t)$ is convex and has at most one minimum. This minimum is either attained on $G^{\mathbb{C}}/G$ or at infinity; the former occurs if and only if $\lim_{t\to\infty} H'_\xi(t) > 0$. So

$$\begin{aligned} &\lim_{t\to\infty} [H'_\xi(t) = -2 \langle \mu_\Omega(z_t), \xi \rangle] > 0 \quad \forall \xi \in \mathfrak{g} - \mathfrak{g}_z \\ &\iff \exists! \text{ minimum of } H_\xi(t) \quad \forall \xi \in \mathfrak{g} - \mathfrak{g}_z \\ &\iff \exists g \in SL(N+1), \mu_\Omega(g \cdot z) = 0 \end{aligned}$$

where $\mathfrak{g}_z$ is the Lie algebra of the stabilizer of $z = Chow(M, L)$, which is $\mathfrak{aut}(M, L)$. This final expression is the condition that (M, L) can be balanced, i.e. $\mu_\Omega(Chow(g \cdot M, L)) = 0$. □

Thus by Donaldson's theorem, Theorem 21 above, asymptotic Chow stability of (M, L) is an obstruction to the existence of a cscK metric on $c_1(L)$ when the stabilizer $Aut(M, L)$ is discrete.

5 K-stability

It has been conjectured by Yau, Tian and Donaldson that another type of stability, K-stability, is equivalent to the existence of cscK metrics. I will describe Tian's proof that the existence of a Kähler-Einstein metric implies weak K-stability, [Tia97].

5.1 Background

The following background is from [Tia00]. K-stability is defined by looking at special degenerations of a Kähler manifold M into normal varieties. Assume $\dim_{\mathbb{C}} M = n \geq 3$.

Definition 37. A fibration is a map $\pi : A \to B$ between two topological spaces which satisfies the homotopy lifting property.

Definition 38. A special degeneration of M is a fibration $\pi : W^{n+1} \to \Delta$, where Δ is the unit disc in $\mathbb{C}$, such that $\pi^{-1}(s)$ is smooth $\forall s \neq 0$, $\pi^{-1}(1/2)$ is biholomorphic to M and there exists $v_W \in \eta(W)$ such that $\pi_* v_W = -s\frac{\partial}{\partial s}$, generating a 1-PS $e^{-s}z$ on Δ. W is trivial if $W = M \times \Delta$.

Tian assumes the central fiber $\pi^{-1}(0)$ and all other fibers are smooth in [Tia00], and defines special degenerations for $\pi^{-1}(0)$ a normal variety in [Tia97]. This more general version of special degeneration is that for which K-stability is defined. Donaldson re-defined K-stability for polarized varieties.

Let W_t denote the fiber $\pi^{-1}(t)$ where $t = e^{-s}$. Since $\pi_* v_W$ vanishes at $t = 0$, v_W restricts to a vector field on W_0, as it has no component in the $\partial/\partial t$ direction on W_0. Thus the Calabi-Futaki invariant $f_{W_0}(v_W) = f_{W_0}(v_W|_{W_0})$ makes sense, using a generalized version of the Futaki invariant when W_0 is normal.

Assume M is a Fano manifold, we've embedded $M \subset \mathbb{CP}^N$ by sections of a power of the anticanonical line bundle, and $W \subset \mathbb{CP}^N \times \Delta$ so $W_t \subset \mathbb{CP}^N \times \{t\}$. Let σ_t be the 1-PS arising from $\Re(v_W)$, $t \in \mathbb{C}$, in the sense that

$$\frac{d}{ds}\sigma_t = \Re(v_W) \tag{24}$$

that is, we have a 1-PS subgroup η_s such that $\frac{d}{ds}\eta_s = \Re(v_W)$, and define $\sigma_t = \sigma_{e^{-s}} := \eta_s$. We have that σ_t flows M to W_t, i.e. $\sigma_t(M) = W_t$ for $t \neq 0$. And $\sigma_t : W_0 \to W_0$ since $\pi_* v_W = 0$ on W_0. All W_t are biholomorphic to M for $t \neq 0$, but the complex structure may "jump" at $t = 0$. For example, when $W_t = \{xy = t\}$, this is not smooth when $t = 0$ but it is when $t \neq 0$.

Since $\sigma_t(W_0) = W_0$ for all $t \in \mathbb{C}$, we can restrict σ_t to W_0 and obtain a 1-parameter subgroup σ_t of diffeomorphisms of $W_0 \subset \mathbb{CP}^N$. So we assume $\sigma_t \in SL(N+1, \mathbb{C})$.

Definition 39. M is weakly K-stable if for every special degeneration W of M

$$\Re(f_{W_0}(v_W)) \geq 0$$

with equality if and only if W is trivial. M is K-stable if it is weakly K-stable and $\eta(M) = \{0\}$.

The main result is:

Theorem 40 ([Tia97])**.** *If a Fano manifold M admits a KE metric, then M is weakly K-stable.*

The proof uses some analytical background. Let $P(M, \omega)$ correspond to the space of Kähler metrics in $[\omega]$,

$$P(M,\omega) = \{\phi \in C^\infty(M, \mathbb{R}) | \omega + \frac{i}{2}\partial\overline{\partial}\phi > 0\}$$

M is a Fano manifold so $\pi c_1(M) = [\omega]$ for some Kähler metric ω. There exists a unique function h_ω s.t.

$$Ric(\omega) - \omega = \frac{i}{2}\partial\overline{\partial}h_\omega$$

$$\frac{1}{V}\int_M e^{h_\omega}\omega^n = 1$$

Suppose M admits a Kähler-Einstein metric ω_ϕ, where $\omega_\phi = \omega + \frac{i}{2}\partial\overline{\partial}\phi$, and assume we've scaled so that $Ric(\omega_\phi) = \omega_\phi$. Then

$$\begin{aligned}
\omega + \frac{i}{2}\partial\overline{\partial}\phi = Ric(\omega_\phi) &= -\frac{i}{2}\partial\overline{\partial}\log\det(\omega_\phi) \\
&= -\frac{i}{2}\partial\overline{\partial}\log\frac{\omega_\phi^n}{\omega^n} + Ric(\omega) \\
&= -\frac{i}{2}\partial\overline{\partial}\log\frac{\omega_\phi^n}{\omega^n} + \omega + \frac{i}{2}\partial\overline{\partial}h_\omega
\end{aligned}$$

which implies

$$\omega_\phi^n = e^{h_\omega - \phi}\omega^n \tag{25}$$

So finding a Kähler-Einstein metric ω_ϕ is equivalent to solving (25). We define a functional F_ω on $P(M, \omega)$

$$F_\omega(\phi) = J_\omega(\phi) - \frac{1}{V}\int_M \phi\omega^n - \log\left(\frac{1}{V}\int_M e^{h_\omega - \phi}\omega^n\right)$$

where $J_\omega(\phi) := \frac{1}{V}\sum_{i=0}^{n-1}\frac{i+1}{n+1}\int_M \partial\phi \wedge \overline{\partial}\phi \wedge \omega^i \wedge \omega_\phi^{n-1-i}$ is called the generalized energy.

Definition 41. F_ω is proper on $P(M,\omega)$ if

1. it is bounded from below, meaning there exists $c = c(\omega) > 0$ such that $F_\omega(\phi) \geq -c$, and
2. there exists an increasing function $\mu : \mathbb{R} \to [c(\omega), \infty)$ such that $\lim_{t\to\infty} \mu(t) = \infty$ and

$$F_\omega(\phi) \geq \mu(J_\omega(\phi))$$

$\forall \phi \in P(M,\omega)$.

The following theorem gives a way of determining if M is KE, when M has no non-trivial holomorphic vector fields.

Theorem 42 ([Tia00, Theorem 6.7])**.** *Assume $\eta(M) = \{0\}$. Then M is Kähler-Einstein if and only if F_ω is proper on $P(M,\omega)$.*

The proof of Theorem 40 makes use of the Sobolev constant, defined as follows.

Definition 43 (Sobolev inequality)**.** Given a Fano manifold (M,ω), there exists a constant σ_ω, called the Sobolev constant of (M,ω), such that $\forall u \in C^\infty(M)$

$$\left(\frac{1}{V}\int_M |u|^{\frac{2n}{n-1}}\omega^n\right)^{\frac{n-1}{n}} \leq \frac{\sigma_\omega}{V}\left(\int_M \partial u \wedge \overline{\partial} u \wedge \omega^{n-1} + \int_M |u|^2\omega^n\right)$$

Definition 44. Define $P(M,\omega,\epsilon)$ to be

$$P(M,\omega,\epsilon) = \{\phi \in P(M,\omega) | \sigma_{\omega_\phi} \leq 1/\epsilon\}$$

where $\omega_\phi = \omega + \frac{i}{2}\partial\overline{\partial}\phi$.

Remark 45 ([Tia97, Example before Thm 5.2]). For a KE Fano manifold (M,ω_{KE}) embedded into $\mathbb{CP}^N$ via K_M^{-k}, the set of $\phi \in P(M,\omega_{KE})$ such that

$$\frac{1}{k}\sigma^*\omega_{FS} = \omega_{KE} + \frac{i}{2}\partial\overline{\partial}\phi, \qquad \text{some } \sigma \in SL(N+1,\mathbb{C})$$

is contained in $P(M,\omega_{KE},\epsilon)$ for some ϵ depending on k and where ω_{FS} is the Fubini-Study metric on $\mathbb{CP}^N$.

Finally, we define K-energy.

Definition 46. Let $\phi \in P(M,\omega)$ and $\{\phi_t\}_{0\leq t\leq 1}$ be any path from 0 to ϕ in $P(M,\omega)$ where $\phi_0 = 0$ and $\phi_1 = \phi$. Then the K-energy of ϕ is

$$\nu_\omega(\phi) = -\frac{1}{V}\int_0^1\int_M \dot{\phi}_t(Ric(\omega_t) - \omega_t) \wedge \omega_t^{n-1} \wedge dt$$

where $\omega_t = \omega + \frac{i}{2}\partial\overline{\partial}\phi_t$ and $\dot{\phi}_t = \frac{\partial\phi_t}{\partial t}$.

The K-energy and the functional F_ω are related by ([Tia00, pg 95])

$$F_\omega(\phi) = \nu_\omega(\phi) + \frac{1}{V}\int_M h_{\omega_\phi}\omega_\phi^n - \frac{1}{V}\int_M h_\omega\omega^n$$

Assuming h_{ω_ϕ} has been normalized so its average value over M is 1 (since $[\omega_\phi] = [\omega] = \pi c_1(M)$ there is some h_{ω_ϕ} such that $Ric(\omega_\phi) - \omega_\phi = \frac{i}{2}\partial\overline{\partial}h_{\omega_\phi}$), we can use the concavity of the logarithmic function to see

$$\begin{aligned}\frac{1}{V}\int_M e^{h_{\omega_\phi}}\omega_\phi^n = 1 \implies 0 &= \log\left(\frac{1}{V}\int_M e^{h_{\omega_\phi}}\omega_\phi^n\right)\\ &\geq \frac{1}{V}\int_M \log\left(e^{h_{\omega_\phi}}\right)\omega_\phi^n\\ &= \frac{1}{V}\int_M h_{\omega_\phi}\omega_\phi^n\end{aligned}$$

Thus if F_ω is proper then ν_ω is too since

$$\nu_\omega(\phi) = F_\omega(\phi) - \frac{1}{V}\int_M h_{\omega_\phi}\omega_\phi^n + \frac{1}{V}\int_M h_\omega \omega^n \tag{26}$$

$$\geq F_\omega(\phi) + \frac{1}{V}\int_M h_\omega\omega^n \tag{27}$$

5.2 Weak K-stability is an obstruction

Sketch proof of Theorem 40, following [Tia97]. In the following assume $t \neq 0$. The Fubini-Study metric on $\mathbb{CP}^N$ restricts to a Kähler metric on W_t. Define $\omega_t = \frac{1}{k}\omega_{FS}|_{W_t}$ where k is such that M is embedded into $\mathbb{CP}^N$ by sections of K_M^{-k}, a power of the anticanonical line bundle. Since $\sigma_t(M) = W_t$, this gives a metric $\tilde{\omega}_t := \sigma_t^*\left(\frac{1}{k}\omega_{FS}|_{W_t}\right)$ on M, which is Kähler as d commutes with pullbacks. Define h_t on W_t by

$$\begin{aligned}Ric(\omega_t) - \omega_t &= \frac{i}{2}\partial\overline{\partial}h_t\\ \implies Ric(\tilde{\omega}_t) - \tilde{\omega}_t &= \frac{i}{2}\partial\overline{\partial}\sigma_t^* h_t\end{aligned}$$

where the h_t are normalized so that their average over W_t is 1.

We've seen that v_W restricts to a vector field on $W_0 \subset \mathbb{CP}^N$. Its real part generates a 1-PS of diffeomorphisms σ_t of W_0. These σ_t are matrices in $SL(N+1, \mathbb{C})$ so give rise to a vector field v on all of $\mathbb{CP}^N$. Since $\overline{\partial}(\iota_v\omega_{FS}) = 0$ (cf the calculation on page 12) there is a smooth θ_v on $\mathbb{CP}^N$ such that

$$\overline{\partial}\theta_v = \frac{1}{k}\iota_v(\omega_{FS})$$

Note that the theorem (F_ω is proper on $P(M,\omega)$) if and only if ((M,ω) is KE) requires $\eta(M) = 0$. Tian gives inequalities which show that F_ω is bounded below when M is KE, but without the assumption $\eta(M) = 0$.

Holomorphic vector fields on a KE manifold M are in one-to-one correspondence with eigenfunctions ψ of the Laplacian of eigenvalue one, i.e. $\Delta\psi = -\psi$. A holomorphic vector field X corresponds to ψ if $g_{KE}(X,Y) = d\psi(Y)$, for any holomorphic vector field Y, where g_{KE} denotes the Kähler-Einstein metric. Let Λ_1 denote the space of eigenfunctions of Δ of eigenvalue one.

Definition 47. We say $\phi \in P(M, \omega_{KE})$ is orthogonal to Λ_1 if

$$\int_M \phi\psi\omega_{KE}^n = 0 \qquad \forall\psi \in \Lambda_1$$

Theorem 48 ([Tia97, Theorem 5.2])**.** *Let (M, ω_{KE}) be a Kähler-Einstein manifold. Then for any $\phi \in P(M, \omega_{KE}, \epsilon)$ with $\epsilon > 0$ and $\phi \perp \Lambda_1$, we have*

$$F_{\omega_{KE}}(\phi) \geq a_{1,\epsilon} J_{\omega_{KE}}(\phi)^{\frac{\beta}{2n+2+\beta}} - a_{2,\epsilon}$$

where $a_{1,\epsilon}, a_{2,\epsilon}$ are constants which depend only on n, ϵ and the lower bound of the difference between the first nonzero eigenvalue of $\Delta_{g_{KE}}$ and 1, i.e. $\lambda_{1,\omega_{KE}} - 1$, from zero. β is a positive constant depending only on n.

Tian shows that it is possible to find suitable automorphisms τ_t of M and $\phi_t \perp \Lambda_1$ in $P(M, \omega_{KE})$, such that

$$\tau_t^* \tilde{\omega}_t = \omega_{KE} + \partial\overline{\partial}\phi_t, \qquad \omega_{KE}^n = e^{h_{\tau_t^*\tilde{\omega}_t} - \phi_t} \tau_t^* \tilde{\omega}_t^n$$

By Remark 45, $\phi_t \in P(M, \omega_{KE}, \epsilon)$, some $\epsilon > 0$. So Theorem 48 applies to $F_{\omega_{KE}}(\phi_t)$.

Next we define a path $\{\psi_s\} \in P(M, \omega_{KE})$, $t = e^{-s}$, which satisfies

$$\tilde{\omega}_t - \omega_{KE} = \partial\overline{\partial}\psi_s, \quad \omega_{KE}^n = e^{h_{\tilde{\omega}_t} - \phi_t} \tilde{\omega}_t^n \tag{28}$$

The K-energy is invariant under automorphisms of M so $\nu_{\omega_{KE}}(\phi_t) = \nu_{\omega_{KE}}(\psi_s)$, ([Tia97]). To determine the rate of change of the K-energy of this path with respect to s, we need $\dot{\psi}_s = \partial\psi_s/\partial s$. Taking the derivative of (28) with respect to s,

$$\frac{\partial\tilde{\omega}_t}{\partial s} = \partial\overline{\partial}\left(\frac{\partial\psi_s}{\partial s}\right) \tag{29}$$

We know σ_t extends to all of $\mathbb{CP}^N$ and $\frac{d}{ds}\sigma_t = \Re(v)$ from (24). Also, by Cartan's formula and page 12 we have $L_X\omega = d(\iota_X\omega) = \partial(\iota_X\omega)$ for a Kähler metric ω and holomorphic vector field X. Then using the Chain rule and the definition of θ_v we have

$$\begin{aligned}
\partial\overline{\partial}\dot{\psi}_s &= \frac{1}{k}\frac{d}{ds}(\sigma_t^*\omega_{FS}) \\
&= \frac{1}{k} L_{\Re(v)}(\sigma_t^*\omega_{FS}) \\
&= \frac{1}{k} d(\iota_{\Re(v)}(\sigma_t^*\omega_{FS})) \\
&= \frac{1}{k} \partial(\iota_{\Re(v)}(\sigma_t^*\omega_{FS})) \\
&= \partial\overline{\partial}\Re(\sigma_t^*\theta_v) \\
\implies \dot{\psi}_s &= \sigma_t^*\Re(\theta_v) + c
\end{aligned}$$

for some constant c. Then if $t(u) = e^{-u}$,

$$\nu_{\omega_{KE}}(\psi_s) = -\frac{1}{V}\int_0^s \int_M \dot{\psi}_u(Ric(\tilde{\omega}_{t(u)}) - \tilde{\omega}_{t(u)}) \wedge \tilde{\omega}_{t(u)}^{n-1} \wedge du$$

$$\begin{aligned}
\therefore \frac{d}{ds}\nu_{\omega_{KE}}(\psi_s) &= -\frac{1}{V}\int_M \dot{\psi}_s(Ric(\tilde{\omega}_{t(s)}) - \tilde{\omega}_{t(s)}) \wedge \tilde{\omega}_{t(s)}^{n-1} \\
&= -\frac{i}{2V}\int_{W_t} \Re(\theta_v)\partial\overline{\partial}h_t \wedge \omega_t^{n-1} \\
&= -\frac{1}{V}\int_{W_t} \Re(\theta_v)\Delta_t h_t \omega_t^n \\
&= \frac{1}{V}\Re\left[-\int_{W_t} g_t^{i\bar{j}}\frac{\partial}{\partial \bar{z}_j}\left(\theta_v \frac{\partial h_t}{\partial z_i}\right)\omega_t^n + \int_{W_t} g_t^{i\bar{j}}\frac{\partial \theta_v}{\partial \bar{z}_j}\frac{\partial h_t}{\partial z_i}\omega_t^n\right] \\
&= \Re\left[\frac{1}{V}\int_{W_t}(\nabla_t\theta_v)h_t\omega_t^n\right]
\end{aligned}$$

where ∇_t denotes the $(1,0)$ gradient with respect to the metric ω_t, $\nabla_t\theta_v = g_t^{i\bar{j}}\frac{\partial\theta_v}{\partial\bar{z}_j}\frac{\partial}{\partial z_i}$. The third line follows from the second by a change of variables from M to W_t and using $Ric(\omega_t) - \omega_t = \frac{i}{2}\partial\overline{\partial}h_t$. The term with c in $\dot{\psi}_s$ vanishes by Stokes' theorem. The penultimate line follows by the chain rule, and the divergence theorem implies the first term in this line vanishes.

In [TD92] Ding and Tian consider deformations of M as $t \to \infty$, converging to some W_∞. This is analogous to the case here, but we're using the parameter t on W_t, where $t = e^{-s}$, instead of W_s. We can apply a result from that paper because $s \to +\infty$ implies W_t converges to W_0. The corresponding result is

$$\lim_{t\to 0}\frac{1}{V}\int_{W_t}(\nabla_t\theta_v)h_t\omega_t^n = f_{W_0}(v_W)$$

Thus

$$\begin{aligned}
\lim_{s\to\infty}\frac{d}{ds}\nu_{\omega_{KE}}(\psi_s) &= \Re\left(\lim_{t\to 0}\left[\frac{1}{V}\int_{W_t}(\nabla_t\theta_v)h_t\omega_t^n\right]\right) \\
&= \Re(f_{W_0}(v_W)) \geq 0
\end{aligned}$$

For if $\frac{d}{ds}\nu_{\omega_{KE}}(\psi_s) < 0$ for all sufficiently large s, then $\nu_{\omega_{KE}}$ becomes arbitrarily negative, contradicting that $\nu_{\omega_{KE}}$ is non-negative when M admits a KE metric ([Tia00, Theorem 7.13]).

Showing that equality occurs if and only if W is trivial uses the following two results proved by Tian.

Theorem 49 ([Tia97, Lemma 6.1]). *Assume W is non-trivial. Then $||\phi_t||_{C^0} \to \infty$ as $t \to 0$.*

Further, the generalized energy $J_{\omega_{KE}}(\phi_t)$ dominates $||\phi_t||_{C^0}$ ([Tia97]) so that $J_{\omega_{KE}}(\phi_t) \to \infty$ as $t \to 0$ by Theorem 49. Since M admits a Kähler-Einstein metric, $F_{\omega_{KE}}(\phi_t) \to \infty$ by Theorem 48 and so $\nu_{\omega_{KE}}(\phi_t) \to \infty$ as $t \to 0$ from Equation 27.

Proposition 50 ([Tia97, Prop 6.2]). *There are positive numbers C, γ which may depend on W such that*

$$\left|\frac{1}{V}\int_{W_t}(\nabla_t\theta_v)h_t\omega_t^n - f_{W_0}(v_W)\right| \leq C|t|^\gamma \tag{30}$$

for small t.

In particular if $f_{W_0}(v_W) = 0$ then

$$\left| \frac{d}{ds} \nu_{\omega_{KE}}(\psi_s) \right| \leq Ce^{-s\gamma}$$

thus $\nu_{\omega_{KE}}(\phi_t) = \nu_{\omega_{KE}}(\psi_s)$ is bounded as $s \to +\infty$. So W must be trivial by Theorem 49.

Conversely if W is trivial, then W_0 is biholomorphic to M which is KE, so W_0 admits a KE metric and the Calabi-Futaki invariant f_{W_0} must vanish.

□

So if one can find a non-trivial special degeneration $\pi : W \to \Delta$ of a Fano manifold M with $f_{W_0} \equiv 0$, then M cannot admit a KE metric.

6 Conclusion

The Kähler-Einstein problem is solved for $c_1(M) \leq 0$ and Fano surfaces. In the general Fano manifold case there are conditions necessary for the existence of KE metrics. As mentioned above these include the Calabi-Futaki invariant, asymptotic Chow stability and weak K-stability. K-stability appears the closest to providing a sufficient condition for the existence of cscK metrics. Donaldson has proved this converse direction in certain cases (with a re-defined notion of K-stability for polarized manifolds) and has nearly solved it in general (as mentioned in discussions with my essay advisor). These metrics give an additional structure to complex manifolds, and the results previously mentioned help establish which manifolds are sufficiently "nice" that they admit KE metrics, or more generally cscK metrics.

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