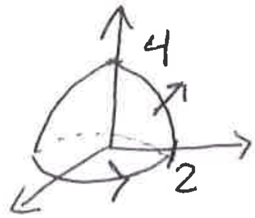


Practice Problems: Stokes' theorem

1) Let S be the portion of the paraboloid $z = 4 - x^2 - y^2$ above the plane $z = 0$, with upward normal. Let $\vec{F} = \langle y - z, -(x+z), x+y \rangle$. Compute $\int\int_S \operatorname{curl} \vec{F} \cdot d\vec{S}$.

directly $\quad \hat{n}\, dS = \oplus \langle 2x, 2y, 1\rangle\, dx\, dy \quad (\hat{k} \text{ cmpt} > 0)$

$$\operatorname{curl}\vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ y-z & -x-z & x+y \end{vmatrix} = \langle 1+1, -1-1, -1-1\rangle = \langle 2, -2, -2\rangle$$

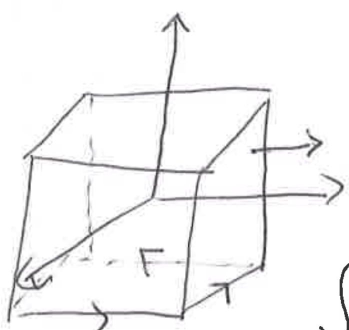
$$\iint_{x^2+y^2 \le 2} \langle 2,-2,-2\rangle \cdot \langle 2x, 2y, 1\rangle\, dx\, dy = \boxed{-8\pi}$$

Stokes $\quad \int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \langle 2\sin t, -2\cos t, 2\cos t - 2\sin t\rangle \cdot \langle -2\sin t, 2\cos t, 0\rangle\, dt$

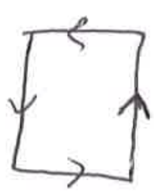
C ccw when viewed from above $\quad \vec{r}(t) = \langle 2\cos t, 2\sin t, 0\rangle \quad = \int_0^{2\pi} -4\, dt = \boxed{-8\pi}$

2) Find $\int\int_S \operatorname{curl} \vec{F} \cdot d\vec{S}$ where $\vec{F} = (xyz)\,\hat{\mathrm{i}} + (xy)\,\hat{\mathrm{j}} + (x^2yz)\,\hat{\mathrm{k}}$ and S consists of the top and 4 sides (no bottom) of the cube with vertices $(\pm 1, \pm 1, \pm 1)$ oriented outward.

Stokes



C = boundary of S is square on bottom of cube.

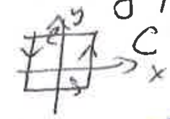
C = 

in $z = -1$ plane

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \langle -xy, xy, -x^2y\rangle \cdot \langle dx, dy, 0\rangle$$

$$= \int_{C'} \langle -xy, xy\rangle \cdot \langle dx, dy\rangle$$

where C' is C in the xy-plane

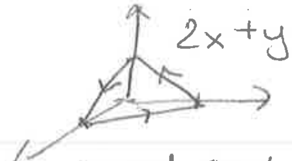


Green's thm (ie Stokes' in 2D)

$$= \int_{-1}^{1}\int_{-1}^{1} \frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(-xy)\, dx\, dy$$

$$= \int_{-1}^{1}\int_{-1}^{1} y + x\, dx\, dy = \boxed{0}$$

The point is that $\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}$ only depends on $\vec{F}$ on the boundary curve C, & both S, & the bottom of the box have same boundary. So do surface integral over bottom face.



3) Let C be the triangle in $\mathbb{R}^3$ with vertices $(1,0,0)$, $(0,2,0)$ and $(0,0,1)$. Compute

$$\int_C (x^2+y)\,dx + yz\,dy + (x-z^2)\,dz \quad = \int_C \vec{F}\cdot d\vec{r}$$

$\vec{F} = \langle x^2+y,\ yz,\ x-z^2\rangle$, $\text{curl}\,\vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ x^2+y & yz & x-z^2 \end{vmatrix}$

$= \langle -y, -1, -1\rangle$

$\int_C \vec{F}\cdot d\vec{r} = \iint_S \langle -y,-1,-1\rangle \cdot \hat{n}\,dS$, $\hat{n}\,dS = \pm\langle -f_x, -f_y, 1\rangle\,dx\,dy$

where $z = f = 1 - x - \frac{1}{2}y$

so upward $\hat{n} \Rightarrow$

$\hat{n}\,dS = \langle 1, \frac{1}{2}, 1\rangle\,dx\,dy$

$= \iint\left(-y - \frac{1}{2} - 1\right)dx\,dy = \int_0^1\int_0^{2-2x} -y - \frac{3}{2}\,dy\,dx = \boxed{-13/6}$

4) Use Stokes' Theorem to evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}(x,y,z) = x^2z\,\hat{\mathbf{i}} + xy^2\,\hat{\mathbf{j}} + z^2\,\hat{\mathbf{k}}$ and C is the curve of intersection of the plane $x+y+z=1$ and the cylinder $x^2+y^2=9$ oriented counterclockwise as viewed from above.

$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$ $\quad$ $\text{curl}\,\vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ x^2z & xy^2 & z^2 \end{vmatrix} = \langle 0, x^2, y^2\rangle$

$\hat{n}\,dS = \pm\langle -f_x, -f_y, 1\rangle\,dx\,dy$

$f = 1 - x - y$

$= \oplus\langle 1,1,1\rangle\,dx\,dy$, $x^2+y^2 \le 9$

$\text{curl}\,\vec{F}\cdot\hat{n}\,dS = (x^2+y^2)\,dx\,dy = r^3\,dr\,d\theta$

$\Rightarrow \int_0^{2\pi}\int_0^3 r^3\,dr\,d\theta = (2\pi)\left(\frac{3^4}{4}\right) = \boxed{\frac{81\pi}{2}}$

Answers: 1) -8π. 2) 0. 3) $-13/6$. 4) $81\pi/2$.