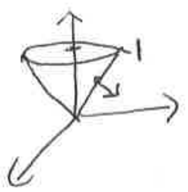
S closed, outward orientation

div thm $\quad \iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \, dV \quad$ where $\partial E = S$

Practice Problems: Divergence theorem, flux integrals

1) Use the divergence theorem to find the flux of $\overrightarrow{F} = (e^z + y^2 x)\,\hat{\mathbf{i}} + (\cos x + x^2 z)\,\hat{\mathbf{k}}$ through the surface S bounded by the cone $z^2 = x^2 + y^2$ and the plane $z = 1$, with outward orientation.



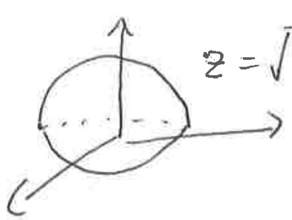
$\operatorname{div} \vec{F} = y^2 + 0 + x^2 = r^2$ in cyl coords

flux of $\vec{F}$ out of S $= \displaystyle\int_0^{2\pi}\int_0^1\int_{z_{min}=r}^{z_{max}=1} r^2 \cdot r \, dz\, dr\, d\theta$

$= (2\pi) \displaystyle\int_0^1 r^3(1-r)\, dr = (2\pi)\left[\left(\tfrac{1}{4}\right) - \left(\tfrac{1}{5}\right)\right]$

$= \dfrac{2\pi}{20} = \boxed{\dfrac{\pi}{10}}$

2) Use the divergence theorem to calculate the flux of $\overrightarrow{F} = |\overrightarrow{r}|\overrightarrow{r}$ through the surface S given by the hemisphere $z = \sqrt{1 - x^2 - y^2}$ and the disk $x^2 + y^2 \leq 1$ in the xy-plane, with outward orientation. (Here $\overrightarrow{r}$ denotes the radial vector $\langle x, y, z \rangle$).



$z = \sqrt{1 - x^2 - y^2}$

$\vec{F} = \sqrt{x^2+y^2+z^2}\,\langle x, y, z \rangle$

$\operatorname{div} \vec{F} = \dfrac{1}{|\vec{r}|}\left(2x^2+y^2+z^2 + x^2+2y^2+z^2 + x^2+y^2+2z^2\right) = 4|\vec{r}|$

$\dfrac{\partial}{\partial x}\left(x\sqrt{x^2+y^2+z^2}\right) = \sqrt{x^2+y^2+z^2} + \dfrac{x^2}{\sqrt{x^2+y^2+z^2}} = \dfrac{2x^2+y^2+z^2}{\sqrt{x^2+y^2+z^2}}$

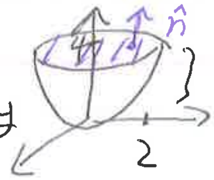
In sphericals, $\operatorname{div} \vec{F} = 4\rho$.

$\displaystyle\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 (4\rho) \cdot \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$

$= 4 \cdot (2\pi)\left(\tfrac{1}{4}\right)\underbrace{\left[-\cos\phi\right]_0^{\pi/2}}_{=1} = \boxed{2\pi}$

3) Use the divergence theorem to evaluate the flux integral $\int\int_S \vec{F}\cdot d\vec{S}$ where $\vec{F} = \langle yz, x^2+y, z^2\rangle$ and S is the part of the paraboloid $z = x^2+y^2$ that lies inside the cylinder $x^2+y^2=4$, with outward orientation. (Note S is not closed so you have to make it closed and subtract out the flux integral over the surface you added in.)

E = solid bdd by



not closed, so add disk on top

disk D at $z=4$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{D+S} \vec{F}\cdot d\vec{S} - \iint_D \vec{F}\cdot d\vec{S}$$

D+S ← closed outward orient^n ⇒ can apply div thm

$$= \iiint_E \underbrace{\operatorname{div}\vec{F}}_{=\,0+1+2z}\, dV - \iint_D \vec{F}\cdot d\vec{S} = \boxed{\iiint_E (1+2z)\, dV} - \iint_D 16\, dA$$

$\vec{F}$ on D: $\langle 4y, x^2+y, 16\rangle$, $d\vec{S} = \langle 0,0,1\rangle\, dx\, dy$

← use cylindrical coords

$\iint_D 16\, dA = 16(\text{area of } D) = 16\cdot\pi\cdot(2)^2 = 64\pi$

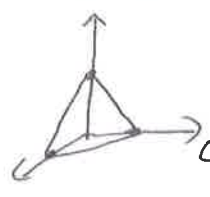
$$\int_0^{2\pi}\int_0^{2}\int_{r^2}^{4} (1+2z)\, r\, dz\, dr\, d\theta = (2\pi)\int_0^2 r\left[z+z^2\right]_{z=r^2}^{z=4} dr$$

$$= (2\pi)\int_0^2 r\left(20 - r^2 - r^4\right) dr = 76\pi/3$$

$$\Rightarrow \text{final answer} = 76\pi/3 - 64\pi = \boxed{-\frac{116\pi}{3}}$$

4) Use the divergence theorem to find the flux of $\vec{F} = y\,\hat{\mathbf{i}} + (z-y)\,\hat{\mathbf{j}} + x\,\hat{\mathbf{k}}$ across the surface S which is the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. Take S to have outward orientation.

$$\operatorname{div}\vec{F} = 0 - 1 + 0 = -1$$



$0 \le x \le 1$
$0 \le y \le 1-x$
$0 \le z \le 1-x-y$

$$\int_0^1\int_0^{1-x}\int_0^{1-x-y} -dz\, dy\, dx = -\int_0^1\int_0^{1-x} 1-x-y\; dy\, dx$$

$$= -\int_0^1 y - xy - \tfrac{1}{2}y^2\Big|_0^{1-x} dx = -\int_0^1 1 - x - x + x^2 - \tfrac{1}{2} - \tfrac{1}{2}x^2 + x\, dx$$

$= \frac{1}{2} - x + \frac{1}{2}x^2$

$$= -\left[\tfrac{1}{2} - \tfrac{1}{2} + \tfrac{1}{6}\right] = \boxed{-1/6}$$

Answers: 1) $\frac{\pi}{10}$. 2) 2π. 4) $-1/6$.