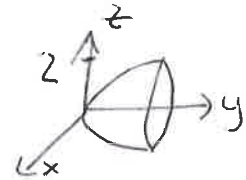


Practice Problems: Flux integrals on surfaces

1) Evaluate the surface integral $\int\int_S y\, dS$ where S is the part of the paraboloid $y = x^2 + z^2$ that lies inside the cylinder $x^2 + z^2 = 4$.

① $\vec{r}(x,z) = \langle x, x^2+z^2, z \rangle, \quad (x,z) \in \{x^2+z^2 \leq 4\}$

② $f(\vec{r}(x,z)) = x^2+z^2$

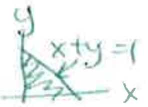
③ $dS = |\vec{r}_x \times \vec{r}_z|\, dx\, dz \quad \begin{vmatrix} i & j & k \\ 1 & 2x & 0 \\ 0 & 2z & 1 \end{vmatrix} = \langle 2x, -1, 2z \rangle \Rightarrow \sqrt{4x^2+4z^2+1}\, dx\, dz = dS$

④ $\displaystyle\iint_{x^2+z^2\leq 4} (x^2+z^2)\sqrt{4x^2+4z^2+1}\, dx\, dz = \int_0^{2\pi}\int_0^2 r^2\sqrt{4r^2+1}\; r\, dr\, d\theta = (2\pi)\cdot \frac{391\sqrt{17}+1}{120}$

$x = r\cos\theta$
$z = r\sin\theta$

integrate by parts w/ $u = r^2,\ v'(r) = r\sqrt{4r^2+1}$

2) Evaluate the flux of $\vec{F}(x,y,z) = xze^y\ \hat{\mathbf{i}} - xze^y\ \hat{\mathbf{j}} + z\ \hat{\mathbf{k}}$ across the surface S consisting of the part of the plane $x+y+z=1$ in the first octant and with downward orientation (meaning $\hat{\mathbf{n}}$ has negative z component).

① $\vec{r}(x,y) = \langle x, y, \underbrace{1-x-y}_{P} \rangle \quad (x,y) \in \{$  $\}$

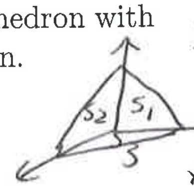
② $\vec{F}(\vec{r}(x,y)) = \langle x(1-x-y)e^y, -x(1-x-y)e^y, 1-x-y \rangle$

③ $\vec{n}\, dS = \pm\langle -f_x, -f_y, 1\rangle\, dx\, dy = \pm\langle 1,1,1\rangle\, dx\, dy$ want negative $= \langle -1,-1,-1\rangle\, dx\, dy$

④ $\displaystyle\int_0^1\int_0^{1-y} \vec{F}(\vec{r}(x,y)) \cdot \langle -1,-1,-1\rangle\, dx\, dy = \int_0^1\int_0^{1-y} -1(\cancel{P}) - 1(\cancel{-P}) - 1(1-x-y)\, dx\, dy$

$\displaystyle= \int_0^1\int_0^{1-y} -1+x+y\, dx\, dy = \boxed{-1/6}$

3) Find the flux of $\vec{F} = y\ \hat{\mathbf{i}} + (z-y)\ \hat{\mathbf{j}} + x\ \hat{\mathbf{k}}$ across the surface S which is the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. Take S to have outward orientation.

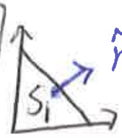


$S_4 =$ front face

$x+y+z=1$

$z = f(x,y) = 1-x-y$

There are 4 surfaces of a tetrahedron.

$\boxed{S_1}$ 

$\hat{n} = \langle -1,0,0\rangle$
$dS = dy\, dz$
b/c a coordinate plane (or parametrize as $\langle 0,y,z\rangle$ and compute $d\vec{S}$)

① $\vec{r}_1(y,z) = \langle 0, y, z\rangle, \quad 0 \leq y \leq 1, \quad 0 \leq z \leq 1-y$

② $\vec{F}(\vec{r}_1(y,z)) = \langle y, z-y, 0\rangle$

③ $d\vec{S} = \langle -1,0,0\rangle\, dy\, dz$

④ $\displaystyle\int_0^1\int_0^{1-y} \langle y, z-y, 0\rangle \cdot \langle -1,0,0\rangle\, dz\, dy = \int_0^1\int_0^{1-y} -y\, dz\, dy$

$\boxed{S_2}$ $\vec{r}_2(x,z) = \langle x, 0, z\rangle$, $\vec{F}(\vec{r}_2(x,z)) \cdot \langle 0,-1,0\rangle = -z$, $\displaystyle\int_0^1\int_0^{1-x} -z\, dz\, dx$

$\boxed{S_3}$ $\displaystyle\int_0^1\int_0^{1-x} \vec{F}(\langle x,y,0\rangle) \cdot \langle 0,0,-1\rangle\, dy\, dx = \int_0^1\int_0^{1-x} -x\, dy\, dx$, $\boxed{S_4}$ $\vec{r}_4(x,y) = \langle x, y, 1-x-y\rangle$

$\vec{F}(\vec{r}_4) = \langle y, 1-x-2y, x\rangle$

$d\vec{S} = \pm\langle -f_x, -f_y, 1\rangle\, dx\, dy$ over →

$$f = 1-x-y$$

$$\vec{F}(\vec{r}_4(x,y)) = \langle y,\ 1-x-2y,\ x\rangle$$

$$d\vec{S} = \pm\langle 1,1,1\rangle\,dx\,dy \qquad \text{(outward so } z\text{-cmpt} > 1\text{)}$$

$$= \langle 1,1,1\rangle\,dx\,dy$$

$$\int_0^1\int_0^{1-y} \underbrace{\langle y,\ 1-x-2y,\ x\rangle\cdot\langle 1,1,1\rangle}_{1-y}\,dx\,dy$$

So let $I = \int_0^1\int_0^{1-y} y\,dx\,dy \ \left(= \int_0^1\int_0^{1-x} y\,dy\,dx\right) = \frac{1}{6}$

$$\iint_S \vec{F}\cdot d\vec{S} = \sum_{i=1}^{4} \iint_{S_i} \vec{F}\cdot d\vec{S}$$

$$= -4I + \underbrace{\int_0^1\int_0^{1-y} dx\,dy}_{\text{area}(\ ) = \frac{1}{2}}$$

$$= -\frac{4}{6} + \frac{1}{2} = \boxed{-1/6}$$

(Check (once we've seen div thm)

$$\iiint_{\text{tetrah}} \operatorname{div}\vec{F}\,dV = \iiint_{\text{tetrah}} -1\,dV = -\text{vol}(\ ) \qquad \text{height} = 1,\ \text{area(base)} = \tfrac{1}{2}$$

$$= -\frac{1}{3}(\text{base}\cdot\text{height})$$

$$= \boxed{-1/6}\ )$$