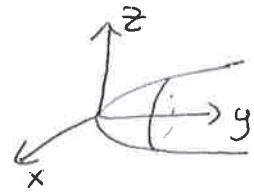
Practice Problems: Surface area for parametric surfaces

1) What surface has vector equation $\vec{r}(s,t) = \langle s\sin(2t), s^2, s\cos(2t)\rangle$? (Find an equation that the coordinates satisfy.)

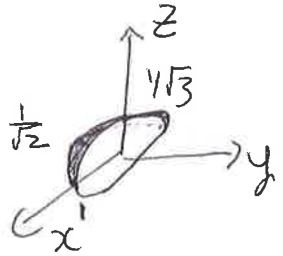
$x^2 + z^2 = y \Rightarrow$ paraboloid along y-axis



whole paraboloid traced out as s and t vary

2) Find a parametric representation for the part of the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$ that lies to the left of the xz-plane. (This is analogous to finding a parametric curve description of the ellipse, but now in 3D we need two angular coordinates.)

Use ϕ and θ but scale lengths:



$$x(\phi,\theta) = \sin\phi\cos\theta$$
$$y(\phi,\theta) = \frac{1}{\sqrt{2}}\sin\phi\sin\theta$$
$$z(\phi,\theta) = \frac{1}{\sqrt{3}}\cos\phi$$

check this satisfies $x^2+2y^2+3z^2=1$

$0 \le \phi \le \pi$, $\pi \le \theta \le 2\pi$ ← only want part on left

3) Find the area of the surface with parametric equations $x = u^2$, $y = uv$, $z = \frac{1}{2}v^2$ for $0 \le u \le 1$ and $0 \le v \le 2$.

① $\vec{r}(u,v) = \langle u^2, uv, \frac{1}{2}v^2\rangle$, $0 \le u \le 1$, $0 \le v \le 2$

② surface area so integrand is 1: $\iint_S dS$

③ $dS = |\vec{r}_u \times \vec{r}_v|\,du\,dv$

$$\begin{vmatrix} i & j & k \\ 2u & v & 0 \\ 0 & u & v \end{vmatrix} = \langle v^2, -2uv, 2u^2\rangle$$

$$= \sqrt{(2u^2)^2 + (v^2)^2 + 4(uv)^2}$$

④ $$\int_0^2\int_0^1 \underbrace{\sqrt{4u^4+v^4+4u^2v^2}}_{=(2u^2+v^2)^2}\,du\,dv$$

$$= \int_0^2\int_0^1 2u^2+v^2\,du\,dv$$

$$= \boxed{4}$$

ignore → this part

~~Coord change~~ $u = r\cos\theta$, $v = 2r\sin\theta$

$$du\,dv = \left|\frac{\partial(u,v)}{\partial(r,\theta)}\right|dr\,d\theta$$
$$= \left|\det\begin{pmatrix}\cos\theta & -r\sin\theta \\ 2\sin\theta & +2r\cos\theta\end{pmatrix}\right|dr\,d\theta$$
$$= 2r\,dr\,d\theta$$

but square in polars hard...

4) Set up the integral for finding the area of the part of the surface $y = 4x + z^2$ that lies between the planes $x = 0$, $x = 1$, $z = 0$, $z = 1$.

① $y = f(x,z) = 4x + z^2$

$\vec{r}(x,z) = \langle x, f(x,z), z \rangle$

$0 \le x \le 1, \; 0 \le z \le 1$

② integrate 1 for area

③ $dS = |\vec{r}_x \times \vec{r}_z| \, dx\,dz = \sqrt{1 + f_x^2 + f_z^2}\, dx\,dz$

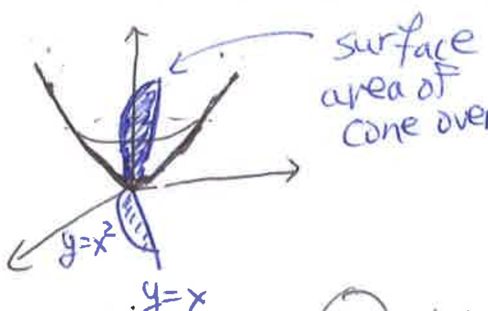
here $f = 4x + z^2$

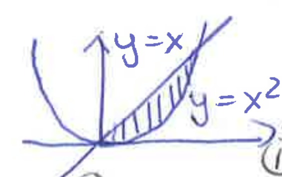
$\begin{vmatrix} i & j & k \\ 1 & f_x & 0 \\ 0 & f_z & 1 \end{vmatrix} = \langle f_x, -1, f_z \rangle$

$\Rightarrow dS = \sqrt{1 + 4^2 + 4z^2}\, dx\,dz$

④ $\iint_S dS = \boxed{\int_0^1 \int_0^1 \sqrt{17 + 4z^2}\, dx\,dz}$

5) Set up the integral for the surface area of the part of the cone $z = \sqrt{x^2 + y^2}$ that lies between the plane $y = x$ and the cylinder $y = x^2$.





① Parametrize, e.g.

(i) $\vec{r}(x,y) = \langle x, y, \sqrt{x^2+y^2} \rangle$ or

(ii) $\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, r \rangle$

(i) bounds $0 \le x \le 1, \; x^2 \le y \le x$

(ii) bounds $0 \le \theta \le \pi/4, \; 0 \le r \le \frac{\sin\theta}{\cos^2\theta}$

② integrate 1

③ (i) $dS = \sqrt{1 + f_x^2 + f_y^2}\, dx\,dy = \sqrt{1 + \frac{x^2+y^2}{x^2+y^2}}\, dx\,dy = \sqrt{2}\, dx\,dy$

(ii) dS: $\begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \langle -r\cos\theta, -r\sin\theta, r \rangle \Rightarrow dS = \sqrt{2}\, r\,dr\,d\theta$

④ integrate (i) or (ii), $\boxed{1/3\sqrt{2}}$

1) paraboloid along y-axis. 2) $(\sin\phi\cos\theta, \frac{1}{\sqrt{2}}\sin\phi\sin\theta, \frac{1}{\sqrt{3}}\cos\phi)$, $0 \le \phi \le \pi$, $\pi \le \theta \le 2\pi$. 3) 4. 4) $\int_0^1\int_0^1 \sqrt{17+4z^2}\,dxdz = \int_0^1 \sqrt{17+4z^2}\,dz$. 5) Parametrize as graph of a function over region bounded by $y = x$ and $y = x^2$ in xy-plane.