Practice Problems: Curl, divergence, flux

1i) Find the curl of $\vec{F}(x,y) = \sin(xy)\hat{\imath} + \cos(xy)\hat{\jmath}$.

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \partial/\partial x & \partial/\partial y \\ \sin(xy) & \cos(xy) \end{vmatrix} = \boxed{-y\sin(xy) - x\cos(xy)}$$

ii) Find the curl of $\vec{F}(x,y,z) = xy\,\hat{\imath} + x^2z\,\hat{\jmath} - (y+z^3)\,\hat{k}$. (note ∂_x means $\partial/\partial x$)

$$\text{Curl } \vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ xy & x^2z & -(y+z^3) \end{vmatrix} = \left\langle \frac{\partial}{\partial y}(-y-z^3) - \frac{\partial}{\partial z}(x^2z), \frac{\partial}{\partial z}(xy) - \frac{\partial}{\partial x}(-y-z^3), \frac{\partial}{\partial x}(x^2z) - \frac{\partial}{\partial y}(xy) \right\rangle$$

$$= \boxed{\langle -1-x^2, 0, 2xz-x \rangle}$$

2) Calculate the divergence of the vector field obtained in 2ii).

$$\text{div}(\langle -1-x^2, 0, 2xz-x \rangle) = \nabla \cdot \langle -1-x^2, 0, 2xz-x \rangle$$

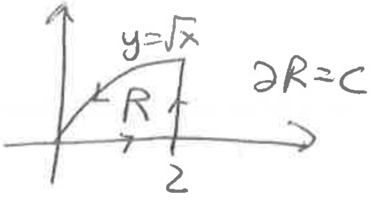
$$= \frac{\partial}{\partial x}(-1-x^2) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(2xz-x)$$

$$= -2x + 2x = \boxed{0}$$

In general

$$\text{curl}(\nabla f) = 0$$

$$\text{div curl } \vec{F} = 0$$



3) Let R be the region enclosed by the x-axis, $x = 2$, and $y = \sqrt{x}$.

(a) Use Green's theorem to compute the flux $\oint_C \vec{F} \cdot \hat{n} ds$ of $\vec{F} = xy\,\hat{\mathbf{i}}$ out of R.

$$\oint_C \vec{F}\cdot\hat{n}\,ds = \iint_R \operatorname{div}\vec{F}\,dA = \iint_R \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(0)\,dA$$

$$= \int_0^2\int_0^{\sqrt{x}} y\,dy\,dx = \frac{1}{2}\int_0^2 x\,dx = \frac{1}{4}\cdot 4 = \boxed{1}$$

(b) Find the flux of $\vec{F}$ out of R through the two segments C_1 (horizontal) and C_2 (vertical).

$\underline{C_1}$ $\vec{r}(t) = \langle t, 0\rangle$, $\hat{n}ds = \langle y'(t), -x'(t)\rangle dt = \langle 0,-1\rangle dt$

$\vec{F}(\vec{r}(t)) = \langle t\cdot 0, 0\rangle = \langle 0,0\rangle \Rightarrow \int_0^2 \langle 0,0\rangle\cdot\langle 0,-1\rangle dt = \boxed{0}$

$\underline{C_2}$ $\vec{r}(t) = \langle 2, t\rangle$, $\hat{n}ds = \langle 1, 0\rangle dt$

$\vec{F}(\vec{r}(t)) = \langle 2t, 0\rangle \Rightarrow \int_0^{\sqrt{2}} \langle 2t,0\rangle \overset{2t}{\overset{\parallel}{\cdot}} \langle 1,0\rangle dt = \boxed{2}$

(c) Using (a) and (b), find the flux out of the third side C_3.

$$\underset{\sum_{i=1}^{3}\int_{C_i}\vec{F}\cdot\hat{n}ds}{\underset{\parallel}{\oint_C \vec{F}\cdot\hat{n}ds}} - \left(\int_{C_1}\vec{F}\cdot\hat{n}ds + \int_{C_2}\vec{F}\cdot\hat{n}ds\right) = \int_{C_3}\vec{F}\cdot\hat{n}ds$$

$$= 1 - 0 - 2 = \boxed{-1}$$

Answers: 1i) $-y\sin(xy) - x\cos(xy)$, ii) $-(1+x^2)\hat{\imath} + (-x+2xz)\hat{k}$. 2) 0. 3a) 1, b) 0 and 2, c) -1