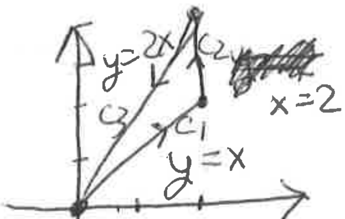
Practice Problems: Green's theorem

1) Evaluate the following two integrals (a) directly and (b) using Green's theorem.

(i) $\int_C xy^2\,dx + 2x^2y\,dy$ where C is the triangle with vertices $(0,0)$, $(2,2)$ and $(2,4)$.



(a) $\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3}$

C_1 $\vec{r}(t) = \langle t, t\rangle$ $\quad 0 \le t \le 2$

$\int_0^2 3t^3\,dt = \frac{3}{4}\cdot 16 = 12$

C_2 $\vec{r}(t) = \langle 2, t\rangle$, $2 \le t \le 4 \Rightarrow \int_2^4 2^3 t\,dt = \frac{8}{2}(16-4) = 48$

C_3 $\vec{r}(t) = \langle t, 2t\rangle$, t from 2 to 0; $\int_2^0 4t^3\,dt + 2^3t^3\,dt = -\int_0^2 12t^3\,dt = -3\cdot 16 = -48$

$\Rightarrow \int_C = 12 + 48 - 48 = \boxed{12}$

(b) $\int_0^2 \int_x^{2x} (\underbrace{4xy}_{2xy} - 2xy)\,dy\,dx = \int_0^2 xy^2\Big|_x^{2x}\,dx = \int_0^2 3x^3\,dx = \frac{3}{4}\cdot 16 = \boxed{12}$

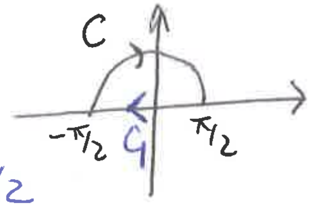
(ii) $\oint_C (x-y)\,dx + (x+y)\,dy$ where C is the circle centered at the origin of radius 2.

$C: \vec{r}(t) = \langle 2\cos t, 2\sin t\rangle$

a) $4\int_0^{2\pi} (\cos t - \sin t)(-\sin t) + (\cos t + \sin t)(\cos t)\,dt = 4\int_0^{2\pi} \cos^2 t + \sin^2 t\,dt = \boxed{8\pi}$

b) $\iint_{\substack{\text{disc}\\ x^2+y^2\le 4}} \frac{\partial}{\partial x}(x+y) - \frac{\partial}{\partial y}(x-y)\,dA = \iint_{\text{disc}} 1+1\,dA = 2\cdot\text{area}(\; 2 \;)$

$= 2\cdot\pi(2)^2 = \boxed{8\pi}$



$-C_1$:
$\vec{r}(t) = \langle t, 0 \rangle$
$-\pi/2 \le t \le \pi/2$
$\vec{r}\,'(t) = \langle 1, 0 \rangle$

$C \cup C_1$ is negatively oriented (clockwise)
$-(C \cup C_1)$ is positively " (counter clockwise) (CCW)
and closed

2) Use Green's theorem to evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x,y) = \langle e^{-x} + y^2, e^{-y} + x^2 \rangle$ and C is the arc of the curve $y = \cos x$ from $(-\pi/2, 0)$ to $(\pi/2, 0)$.

Can apply Green's thm to $\displaystyle\int_{-(C \cup C_1)} \vec{F} \cdot d\vec{r} = \int_{-\pi/2}^{\pi/2} \int_0^{\cos x} (2x - 2y)\, dy\, dx$

0 b/c $x\cos x$ odd

0 b/c $\sin\pi = 0$

$$= \int_{-\pi/2}^{\pi/2} 2x\cos x - \cos^2 x \, dx = \underbrace{2x \cdot \sin x \Big|_{-\pi/2}^{\pi/2}}_{0} - \int_{-\pi/2}^{\pi/2} 2\sin x \, dx - \frac{1}{2}\int_{-\pi/2}^{\pi/2} 1 + \cos(2x)\, dx$$

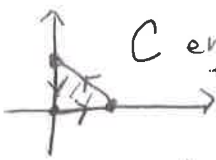
$\hookrightarrow$ 0 b/c sin odd

$$= -\pi/2 \Rightarrow \int_C \vec{F} \cdot d\vec{r} = -\int_{-C} \vec{F} \cdot d\vec{r} = -\left[\int_{-(C \cup C_1)} \vec{F} \cdot d\vec{r} - \int_{-C_1} \vec{F} \cdot d\vec{r} \right]$$

$$\int_{-C_1} \vec{F} \cdot d\vec{r} = \int_{-\pi/2}^{\pi/2} e^{-t}\, dt$$

$$= -\left(-\frac{\pi}{2} - [e^{\pi/2} - e^{-\pi/2}]\right) = \boxed{\pi/2 + e^{\pi/2} - e^{-\pi/2}}$$

3) Use Green's theorem to find the work done by the force $\vec{F}(x,y) = x(x+y)\hat{\imath} + xy^2\hat{\jmath}$ in moving a particle from the origin along the x-axis to $(1,0)$, then along the line segment to $(0,1)$, and then back to the origin along the y-axis.



C encloses triangle

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_{\triangle} \frac{\partial}{\partial x}(xy^2) - \frac{\partial}{\partial y}(x^2 + xy)\, dA$$

$$= \int_0^1 \int_0^{1-x} (y^2 - x)\, dy\, dx = \int_0^1 \frac{1}{3}y^3 - xy \Big|_0^{1-x} dx$$

$$= \int_0^1 \frac{1}{3}(1-x)^3 - x + x^2\, dx$$

$$= -\frac{1}{12}(1-x)^4 \Big|_0^1 - \frac{1}{2} + \frac{1}{3}$$

$$= \frac{1}{12} - \frac{6}{12} + \frac{4}{12} = \boxed{-\frac{1}{12}}$$

Answers: 1) i) 12, ii) 8π. 2) Use Green's thm to evaluate integral around a closed curve, then $\int_C = \int_{closed\ curve} - \int_{line\ segment}$. Answer: $\pi/2 + e^{\pi/2} - e^{-\pi/2}$. 3) $-1/12$.