Practice Problems: fundamental theorem of line integrals, conservative vector fields

1) Determine if $\vec{F}$ is conservative and if so, find its scalar potential function f, i.e. $\vec{F} = \nabla f$.

(a) $\vec{F}(x,y) = e^x \cos y\hat{\imath} + e^x \sin y\hat{\jmath}$. $\frac{\partial}{\partial x}(e^x \sin y) = e^x \sin y \neq \frac{\partial}{\partial y}(e^x \cos y)$ so $\boxed{\text{no}}$

$= -e^x \sin y$

(b) $\vec{F}(x,y) = (\ln y + 2xy^3)\hat{\imath} + (3x^2y^2 + x/y)\hat{\jmath}$.

$\underset{P}{\shortparallel}$ $\underset{Q}{\shortparallel}$

$$\left.\begin{aligned} \frac{\partial Q}{\partial x} &= 6xy^2 + \frac{1}{y} \\ \frac{\partial P}{\partial y} &= 6xy^2 + \frac{1}{y} \end{aligned}\right\}$$ valid on $y>0$ a simply connected region

$$\int P\,dx = x\ln y + x^2y^3 + g(y)$$

$$\frac{\partial f}{\partial y} = \frac{x}{y} + 3x^2y^2 + g'(y) = 3x^2y^2 + \frac{x}{y} \Rightarrow g'(y) = 0$$

$$\therefore g(y) = C$$

$$\Rightarrow \boxed{f = x\ln y + x^2y^3 + C \text{ on } y>0}$$

2) Find a function f such that $\vec{F} = \nabla f$, where $\vec{F}(x,y) = xy^2\hat{\imath} + x^2y\hat{\jmath}$.

$$\int P\,dx = \frac{1}{2}x^2y^2 + g(y), \quad \frac{\partial f}{\partial y} = x^2y + g'(y) = x^2y$$

$$\Rightarrow g'(y) = 0 \therefore g(y) = C$$

$$\boxed{f = \frac{1}{2}x^2y^2 + C}$$

$\vec{F} = < xy^2, x^2y >$

3) (a) For the $\overrightarrow{F}$ in the previous problem, evaluate $\int_C \overrightarrow{F} \cdot d\overrightarrow{r}$ along the curve $C : \overrightarrow{r}(t) = \langle t + \sin(\pi t/2), t + \cos(\pi t/2) \rangle$ for $0 \leq t \leq 1$.

(b) Evaluate the same line integral where now C is the straight line from $(0, 1)$ to $(2, 1)$. Note this has the same endpoints as in (b).

(c) Now do the integral using the Fundamental Theorem for line integrals. You should get the same answer in all three cases.

a) $\vec{r}\,'(t) = < 1 + \frac{\pi}{2}\cos(\frac{\pi t}{2}), 1 - \sin(\frac{\pi t}{2}) \cdot \frac{\pi}{2} >$

$\int_C (xy^2)\,dx + (x^2y)\,dy = \int_0^1 (t+\sin(\frac{\pi t}{2}))(t+\cos(\frac{\pi t}{2}))^2(\ )\,dt$

$= 2 \qquad + (t+\sin(\frac{\pi t}{2}))^2(t+\cos(\frac{\pi t}{2}))(\ )\,dt$

b) $\vec{r}(t) = (1-t) <0, 1> + t<2, 1> = <2t, 1>$

$\vec{r}\,'(t) = < 2, 0 >$

$\int_0^1 2t \cdot 2 + 0 \,dt = 4\int_0^1 t\,dt = 4 \cdot \frac{1}{2}t^2\Big|_0^1 = 2$

c) $\int_C \vec{F} \cdot d\vec{r} = f(2,1) - f(0,1)$ recall $f = \frac{1}{2}x^2y^2$ $(+C)$

$= \frac{1}{2} \cdot 4 - 0 = \boxed{2}$

4) Explain why the following holds: Suppose a vector field $\overrightarrow{F}(x, y)$ is perpendicular to the tangent vector $\overrightarrow{r}\,'(t)$ to a curve C, at each point $(x(t), y(t))$ on the curve. Then $\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = 0$.

$\int_C \vec{F} \cdot d\vec{r} = \int_{t_0}^{t_1} \underbrace{\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)}_{0}\,dt = 0.$

Extra line integral practice

5) Evaluate $\int_C (xy + \ln x)dy$ where C is the arc of the parabola $y = x^2$ from $(1, 1)$ to $(3, 9)$.

6) The position of an object with mass m at time t is $\overrightarrow{r}(t) = at^2\hat{\imath} + bt^3\hat{\jmath}$, for $0 \leq t \leq 1$. (a) Find the force acting on the object at time t. (b) What is the work done by the force during $0 \leq t \leq 1$?

Answers: 1) No, Yes, $x \ln y + x^2y^3 + C$. 2) $x^2y^2/2$. 3) 2. 4) dot product in integral is zero along C. 5) $464/5 + 9 \ln 3$. 6) a) Use $\overrightarrow{F} = m\overrightarrow{a}$ to get $2ma\hat{\imath} + 6bmt\hat{\jmath}$, b) $m(2a^2 + \frac{9}{2}b^2)$.