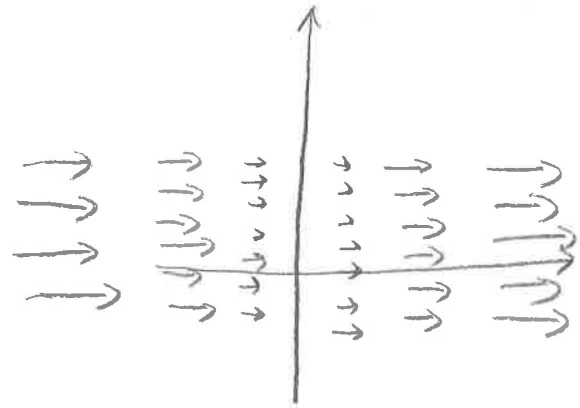
Practice Problems: Vector fields, line integrals of a scalar function

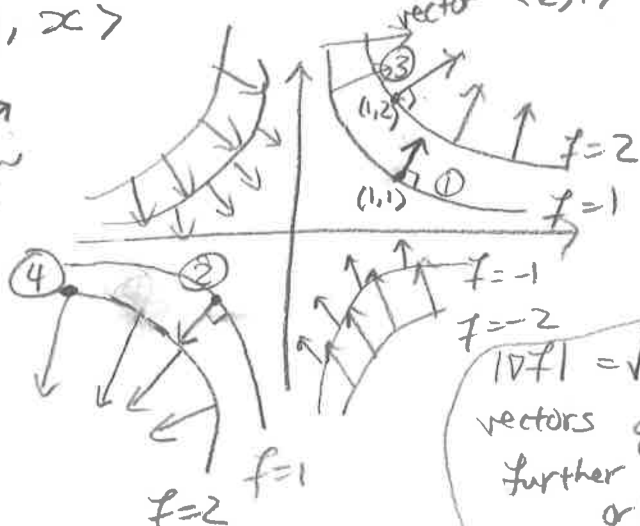
1) Sketch the vector field $\overrightarrow{F}(x,y) = x^2\ \hat{\mathbf{i}}$.



2) Sketch the contour plot and gradient vector field of $f(x,y) = xy$. For example, you could start by sketching the contours $f(x,y) = 1$ and $f(x,y) = 2$.

$\nabla f = \langle y, x \rangle$

vector $\langle 2,1 \rangle$ at $(1,2)$



① $\nabla f\big|_{(1,1)} = \langle 1,1 \rangle$

place vector at $(1,1)$

② $\nabla f\big|_{(-1,-1)} = \langle -1,-1 \rangle$

$|\nabla f| = \sqrt{x^2+y^2}$
vectors get longer further from origin

③ $\nabla f\big|_{(1,2)} = \langle 2,1 \rangle$

at $(1,2)$
put vector $\langle 2,1 \rangle$

④ $\nabla f\big|_{(-2,-1)} = \langle -1,-2 \rangle$

3) Find $\int_C z\, ds$ where C is the helix $(\cos t, \sin t, t)$ for $0 \le t \le \pi$.

$$\int_0^{\pi} t\sqrt{(-\sin t)^2+(\cos t)^2+1}\; dt = \int_0^{\pi} \sqrt{2}\, t\, dt$$

$$= \frac{\pi^2\sqrt{2}}{2}$$

4) Find $\int_C y^2 z\, ds$ where C is the line segment from $(3,1,2)$ to $(1,2,5)$.

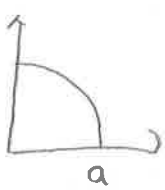
$$\vec{r}(t) = (1-t)\langle 3,1,2\rangle + t\langle 1,2,5\rangle$$

$$= \langle 3-3t+t,\ 1-t+2t,\ 2-2t+5t\rangle$$

$$= \langle 3-2t,\ 1+t,\ 2+3t\rangle$$

$$\int_0^1 (1+t)^2(2+3t)\sqrt{(-2)^2+1^2+3^2}\; dt = \frac{107\sqrt{7}}{6\sqrt{2}}$$

5) A thin wire has the shape of the first-quadrant part of the circle with center the origin and radius a. If the density function is $\rho(x,y) = kxy$, find the mass and center of mass of wire.



$$\vec{r}(t) = \langle a\cos t,\ a\sin t\rangle \qquad ds = |\vec{r}\,'(t)|\, dt$$

$$\vec{r}\,'(t) = \langle -a\sin t,\ a\cos t\rangle \qquad = a\, dt$$

$$m = \int_0^{\pi/2} k\cdot(a\cos t)(a\sin t)\cdot a\, dt = ka^3\int_0^{\pi/2}\cos t\sin t\, dt = \boxed{\frac{ka^3}{2}}$$

$$\bar{x} = \frac{1}{m}\int_C x\rho\, ds = \frac{2}{ka^3}\int_0^{\pi/2} k(a\cos t)^2(a\sin t)\cdot a\, dt = \frac{2a}{3} \qquad \boxed{(\bar{x},\bar{y}) = \left(\frac{2a}{3}, \frac{2a}{3}\right)}$$

kx^2y

$$\bar{y} = \frac{2}{ka^3}\int_0^{\pi/2} k(a\cos t)(a\sin t)^2 a\, dt = \frac{2a}{3}$$

1) arrows point right only, increasing in length with $|x|$. 2) vector field arrows perpendicular to contour lines. 3) $\frac{\pi^2\sqrt{2}}{2}$.