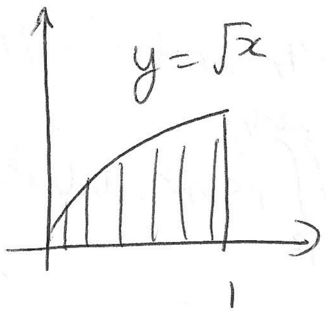
Practice Problems: Applications of triple integrals

1) Find the mass and set up the x-coordinate of the center of mass of the solid E with constant density function $\rho = 2$, where:

E lies under the plane $1 + x + y$ and above the region in the xy-plane bounded by the curves $y = \sqrt{x}$, $y = 0$ and $x = 1$.



$$\text{mass} = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 2 \, dz\, dy\, dx$$

$$= 2\int_0^1 \int_0^{\sqrt{x}} 1+x+y \, dy\, dx$$

$$= 2\int_0^1 y + xy + \frac{1}{2}y^2 \Big|_0^{\sqrt{x}} dx$$

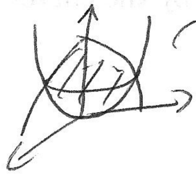
$$= 2\int_0^1 \sqrt{x} + x^{3/2} + \frac{1}{2}x \, dx$$

$$\Rightarrow m = \boxed{79/30}$$

$$\bar{x} = \frac{1}{m} \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 2x \, dz\, dy\, dx$$

2) a) Find the volume of the region E bounded by the paraboloids $z = x^2 + y^2$ and $z = 36 - 3x^2 - 3y^2$.

b) Set up the integrals to find the centroid of E (the center of mass in the case where density is constant.) Explain what $\bar{x}$ and $\bar{y}$ are without doing a calculation.

a) 

$r^2 = 36-3r^2$

$\Rightarrow 4r^2 = 36$

$\Rightarrow r^2 = 9$

$\Rightarrow r = 3$

$$\text{vol} = \int_0^{2\pi}\int_0^3\int_{r^2}^{36-3r^2} 1\cdot r\,dz\,dr\,d\theta$$

$$= (2\pi)\int_0^3 (36-4r^2)\,r\,dr = 162\pi$$

b) $$(\bar{x},\bar{y},\bar{z}) = \left(\frac{1}{162\pi}\int_0^{2\pi}\int_0^3\int_{r^2}^{36-3r^2} r\cos\theta\cdot r\,dz\,dr\,d\theta,\ \frac{1}{162\pi}\int_0^{2\pi}\int_0^3\int_{r^2}^{36-3r^2} r\sin\theta\cdot r\,dz\,dr\,d\theta,\right.$$

$$\left.\frac{1}{162\pi}\int_0^{2\pi}\int_0^3\int_{r^2}^{36-r^2} z\cdot r\,dz\,dr\,d\theta\right)$$

$\bar{x} = \bar{y} = 0$ by symmetry of E & integrand

3) Find the moments of inertia of a cube of constant density k and side length L, if one vertex is located at the origin and three edges lie along the coordinate axes.

density $= k$

$0 \le x, y, z \le L$

$$I_x = \int_0^L\int_0^L\int_0^L k(y^2+z^2)\,dx\,dy\,dz = k\int_0^L\int_0^L L(y^2+z^2)\,dy\,dz$$

$$= \frac{2k}{3}L^5$$

$I_x = I_y = I_z$ by symmetry

1) $m = 79/30$, $\bar{x} = \frac{1}{m}\int_0^1\int_0^{\sqrt{x}}\int_0^{1+x+y} 2x\,dz\,dy\,dx$

2) 162π, $\bar{x}, \bar{y}$ should be 0 by symmetry. $\bar{z} = \frac{1}{162\pi}\int_0^{2\pi}\int_0^3\int_{r^2}^{36-3r^2} zr\,dz\,dr\,d\theta$.

3) $I_x = I_y = I_z = \frac{2k}{3}L^5$