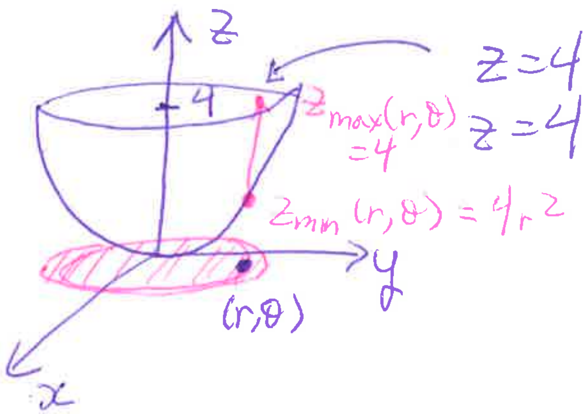
Practice problems: Triple integrals in Cartesian and cylindrical coordinates

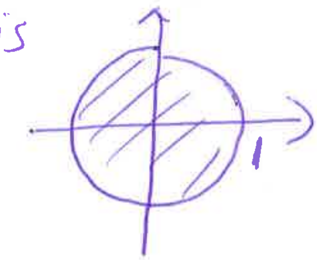
1) Find $\int\int\int_E z\,dV$ where E is the region bounded by the paraboloid $z = 4x^2 + 4y^2$ and the plane $z = 4$.



$z_{max}(r,\theta) = 4$

$z_{min}(r,\theta) = 4r^2$

$\left.\begin{array}{l} z=4 \\ z=4r^2 \end{array}\right\} \Rightarrow r=1$ so ~~[illegible]~~ shadow on xy-plane is



$0 \le r \le 1$

$0 \le \theta \le 2\pi$

$$\text{So} \quad \iiint_E z\,dV = \int_0^{2\pi} \boxed{\int_0^1 \int_{4r^2}^{4} z\,dz\, r\,dr}\,d\theta$$

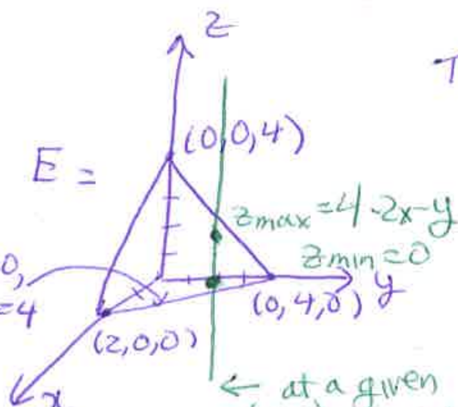
$\longrightarrow$ no dependence on θ in bounds or integrand so can pull out $\int_0^{2\pi} d\theta = 2\pi$. I.e. boxed integral is a constant.

$$= (\cancel{2}\pi)\int_0^1 r\cdot \tfrac{1}{\cancel{2}} z^2 \Big|_{z=4r^2}^{4} dr = \pi\int_0^1 (4^2 - 16r^4)\,r\,dr = \frac{16\pi}{3}$$

2) Find the volume of the tetrahedron enclosed by the coordinate planes and the plane $2x + y + z = 4$. One way to draw the plane is to look at the three points where 2 of the coordinates are zero. (e.g. something similar was done in example in section w/ changing bounds $\int_0^1\int_0^{1-x^2}\int_0^{1-x+z} f\,dy\,dz\,dx$ to order $dz\,dy\,dx$.)

& note 3 pts determine a plane

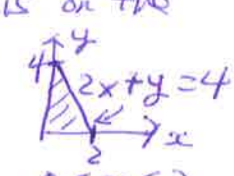
$E =$



Set $z=0$, $2x+y=4$

$\leftarrow$ at a given (x,y) in xy-shadow, line thru (x,y) pierces solid at $z_{min}(x,y)$ & $z_{max}(x,y)$

There are 6 choices for integration: we could do height as a function of x & y, so z is on the inside. Then shadow on xy plane is



and bounds are: ~~[illegible]~~ height goes from 0 to $4-2x$ as x goes from 0 to 2.

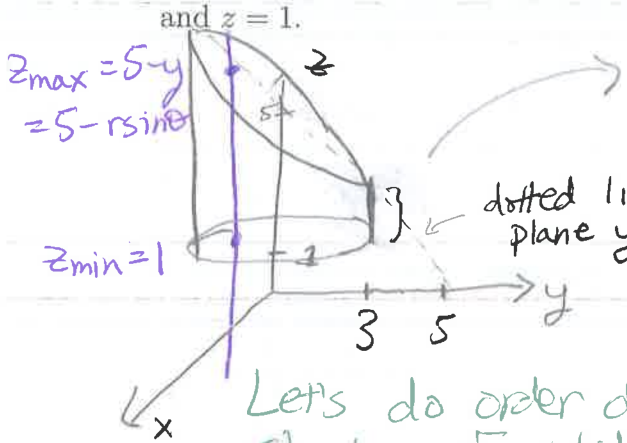
$0 \le x \le 2$

$0 \le y \le 4-2x$

Then at a given (x,y), z goes from 0 to $4-2x-y$. $0 \le z \le 4-2x-y$

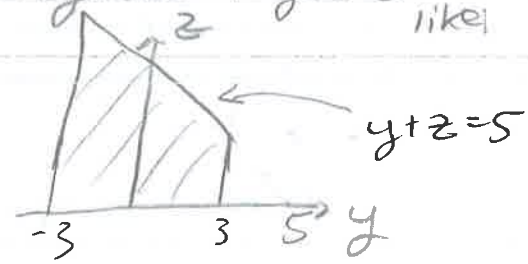
$$\text{vol} = \iiint_E dV = \int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} dz\,dy\,dx = \int_0^{2}\int_0^{4-2x} z\Big|_0^{4-2x-y} dy\,dx = \boxed{16/3}$$

3) Find the volume of the solid enclosed by the cylinder $x^2+y^2=9$ and the planes $y+z=5$ and $z=1$.

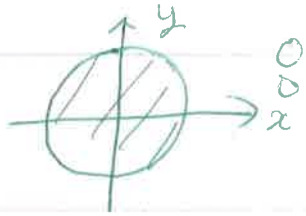


dotted line is plane $y+z=5$ in yz plane

when plane intersects cylinder, it happens at a height bigger than 1. Reason In yz-plane the cylinder & $y+z=5$ look like



Let's do order $dz\,r\,dr\,d\theta$. Shadow of solid in (r,θ) plane? (ie xy plane)



$0 \le r \le 3$
$0 \le \theta \le 2\pi$

Vertical line thru (x,y) pierces solid at $z_{min}(r,\theta)$ and $z_{max}(r,\theta)$:

$$z_{min} = 1$$
$$z_{max} = 5 - r\sin\theta$$

$$\text{vol} = \iiint_{\text{Solid}} 1 \cdot dV$$

$$= \int_0^{2\pi} \int_0^3 \int_1^{5-r\sin\theta} 1 \, dz\, r\, dr\, d\theta$$

A function of θ so can't do θ integral by pulling it out as before.

$$= \int_0^{2\pi}\int_0^3 z\Big]_{z=1}^{z=5-r\sin\theta} r\,dr\,d\theta$$

don't forget the r

$$= \int_0^{2\pi}\int_0^3 (4 - r\sin\theta)\, r\, dr\, d\theta = \boxed{36\pi}$$

1) $16\pi/3$
2) $16/3$
3) 36π