Practice problems: Change of variables in multiple integrals

1) Find the Jacobian of the transformation $x = uv, y = u/v$.

$\dfrac{\partial(x,y)}{\partial(u,v)}$ is the Jacobian (note $dx\,dy = \left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$)

can remember top & bottom by $\partial(u,v)$ "cancels" $du\,dv$ to get $dx\,dy$. Not what's happening mathematically, but way to remember.

$$\det\begin{pmatrix} v & u \\ \frac{1}{v} & \frac{-u}{v^2} \end{pmatrix} = \frac{-uv}{v^2} - \frac{u}{v} = -\frac{2u}{v}$$

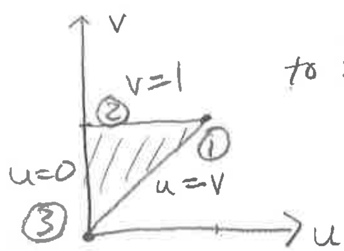
$$\boxed{\frac{\partial(x,y)}{\partial(u,v)} = \frac{-2u}{v}, \quad \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{2u}{v}\right|}$$

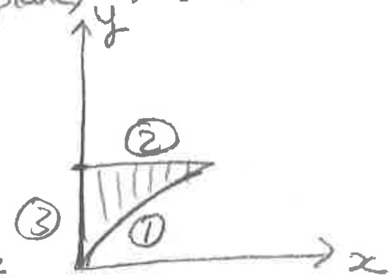
2) Find the image of the set

$S =$ triangular region with vertices $(0,0)$, $(1,1)$, $(0,1)$ in the uv-plane

under the transformation $x = u^2, y = v$.

Note: region is given in uv-plane, & we are transforming to xy-plane.





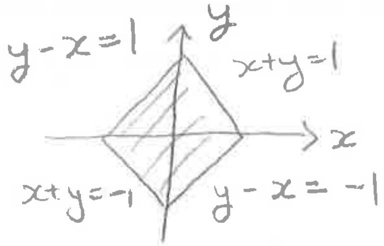
① $u = v \Rightarrow x = y^2$

$0 \le v \le 1 \Rightarrow y = \sqrt{x}$ (not $-\sqrt{x}$, b/c $y = v \geq 0$)

② $y = 1$ (b/c $v = 1$, $0 \le u \le 1$)
$0 \le x \le 1$

③ $u = 0 \Rightarrow x = 0$
$0 \le v \le 1 \quad 0 \le y \le 1$

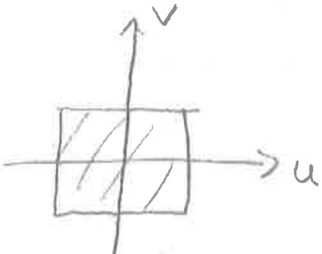
In particular, straight lines don't necessarily go to straight lines.



3) Evaluate $\int\int_R e^{x+y} dA$ over $R = \{|x| + |y| \leq 1\}$ by making a suitable change of variables.

$u = x+y$

$v = x-y$ (b/c region R becomes a square)



$-1 \leq u \leq 1$
$-1 \leq v \leq 1$

$$\int_{-1}^{1}\int_{-1}^{1} e^u \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$= \int_{-1}^{1}\int_{-1}^{1} e^u \tfrac{1}{2}\, du\,dv$$

$$= (1-(-1))\, e^u\Big|_{-1}^{1} \cdot \tfrac{1}{2}$$

$$= \boxed{e - e^{-1}}$$

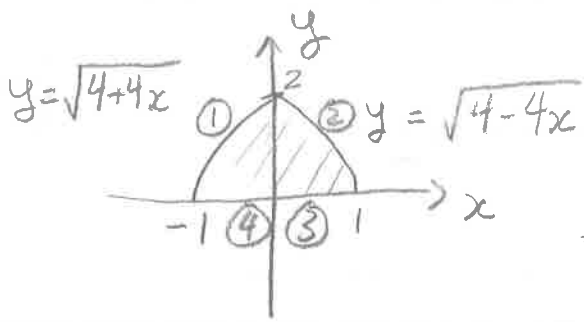
<u>Note</u> u_x means $\partial u/\partial x$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\det\begin{pmatrix} u_x & u_y \\ v_x & v_y\end{pmatrix}}$$

$$= 1\Big/\det\begin{pmatrix} 1 & 1 \\ 1 & -1\end{pmatrix} = -1/2$$

4) Use the change of variables $x = u^2 - v^2$, $y = 2uv$ to evaluate the integral $\int\int_R y\, dA$ where R is the region bounded by the x-axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$, $y \geq 0$.

(assume $u, v \geq 0$ so we have a one-to-one mapping)



① $y = \sqrt{4+4x}$

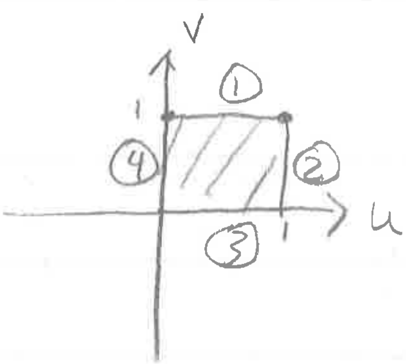
$\Rightarrow 2uv = \sqrt{4 + 4u^2 - 4v^2} \Rightarrow$ u, v same sign b/c $uv \geq 0$. We assume both positive.

$\Rightarrow 4u^2v^2 - 4u^2 + 4v^2 - 4 = 0 \quad \div 4$

$\Rightarrow (u^2+1)(v^2-1) = 0$

$\Rightarrow v = \pm 1 \;\&\; \geq 0 \Rightarrow v = 1$

Then $y = 2uv = 2u \;\&\; 0 \leq y \leq 2$
$\Rightarrow 0 \leq u \leq 1$



② $y = \sqrt{4-4x} \Rightarrow (u^2-1)(v^2+1) = 0 \Rightarrow u = 1$
$\Rightarrow 0 \leq y = 2v \leq 2 \Rightarrow 0 \leq v \leq 1$

③ $y = 0$
④ $\underbrace{}_{u=0 \text{ or } v=0}$

: $u = 0 \Rightarrow x = -v^2$ so corresponds to ④ $0 \leq v \leq 1$
$v = 0 \Rightarrow x = u^2$ —— " —— ③ $0 \leq u \leq 1$

Jacobian: $\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} 2u & -2v \\ 2v & 2u\end{pmatrix} = 4u^2 + 4v^2 \Rightarrow dx\,dy = (4u^2+4v^2)\,du\,dv$

Finally compute:

$$\underbrace{\int_0^1\int_0^1}_{\substack{\text{new}\\ \text{domain}\\ \text{in } u,V}} \underset{\substack{\uparrow\\ \text{integrand}\\ y(u,v)}}{(2uv)} \underbrace{(4u^2+4v^2)\,du\,dv}_{dx\,dy} = 2$$

<u>Note</u> In §15.9 Change of Variables in multiple integrals, they do this problem in examples 1 & 2. There, they show that, given the region in the xy-plane, the transformation $x = u^2 - v^2$, $y = 2uv$ maps the unit square (in u, v) to the given region in x, y.

Here, we went in the other direction, looking at the transformation to the uv-plane.