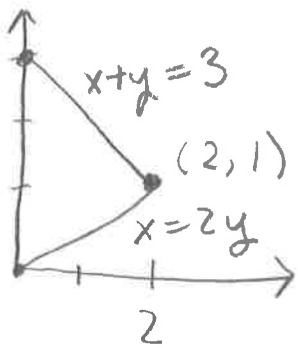
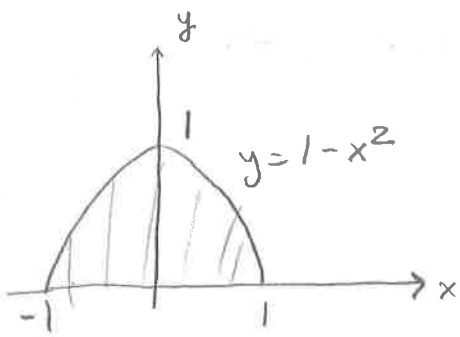
Practice problems: Applications of double integrals

1) Set up the integral for the mass of the lamina that occupies D = triangular region with vertices $(0,0)$, $(2,1)$ $(0,3)$ and has density function $\rho(x,y) = x + y$.



$$\text{mass} = \int_0^2 \int_{\frac{x}{2}}^{3-x} (x+y)\, dy\, dx$$

2) Find the center of mass of the lamina with density function $\rho(x,y) = ky$ that occupies the region D which is bounded by $y = 1 - x^2$ and $y = 0$.



$$\text{mass} = \int_{-1}^{1} \int_0^{1-x^2} (ky)\, dy\, dx$$

$$= \int_{-1}^{1} \frac{k}{2} y^2 \Big|_0^{1-x^2} dx = \frac{k}{2} \int_{-1}^{1} (1-x^2)^2\, dx$$

$$= \frac{k}{2} \int_{-1}^{1} 1 - 2x^2 + x^4\, dx$$

$$= k/2 \left[x - \frac{2}{3}x^3 + \frac{x^5}{5} \right]_{-1}^{1}$$

$$= k/2 \left(2 - 4/3 + 2/5 \right) = \frac{8k}{15}$$

$$-\frac{20}{15} + \frac{6}{15} = -\frac{14}{15}$$

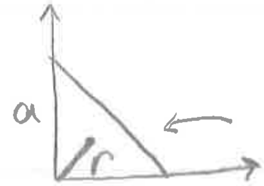
$$\frac{30-14}{15} = \frac{16}{15}$$

$$\bar{x} = \frac{1}{8k/15} \int_{-1}^{1} \int_0^{1-x^2} x\,(ky)\, dy\, dx = 0$$

$$\bar{y} = \frac{1}{8k/15} \int_{-1}^{1} \int_0^{1-x^2} y\,(ky)\, dy\, dx = \frac{4}{7}$$

$$\boxed{(0, 4/7)}$$

3) Find the center of mass of a lamina in the shape of an isosceles right triangle with equal sides of length a, if the density at any point is proportional to the square of the distance from the vertex opposite the hypotenuse. [*By symmetry you only need to find one coordinate in the center of mass.*]



$$\rho(x,y) = k(x^2+y^2)$$

$\leftarrow$ $x+y=a$, $r(\cos\theta+\sin\theta) = a$, $r = \dfrac{a}{\cos\theta+\sin\theta}$

$0 \le \theta \le \pi/2$

(harder to do in polars)

$$\text{mass} = \int_0^{\pi/2}\int_0^{a/\cos\theta+\sin\theta} k r^2 \cdot r\,dr\,d\theta = \int_0^a\int_0^{a-x} k(x^2+y^2)\,dy\,dx = \frac{ka^4}{6}$$

$$\bar{x} = \frac{1}{m}\int_0^a\int_0^{a-x} kx(x^2+y^2)\,dy\,dx = \frac{1}{m}\int_0^a\int_0^{a-y} ky(x^2+y^2)\,dx\,dy = \bar{y}$$

$$= \frac{6}{a^4}\left(\frac{a^5}{15}\right) = \frac{2a}{5} \Rightarrow \boxed{(2a/5,\ 2a/5)}$$

4) Set up the integrals for the moments of inertia I_0, I_x, I_y in the previous problem.

$$I_0 = \int_0^a\int_0^{a-x} k(x^2+y^2)^2\,dy\,dx \qquad \left(= \iint_D (x^2+y^2)\rho\,dA\right)$$

$$I_x = \int_0^a\int_0^{a-x} ky^2(x^2+y^2)\,dy\,dx$$

$$I_y = \int_0^a\int_0^{a-x} kx^2(x^2+y^2)\,dy\,dx$$