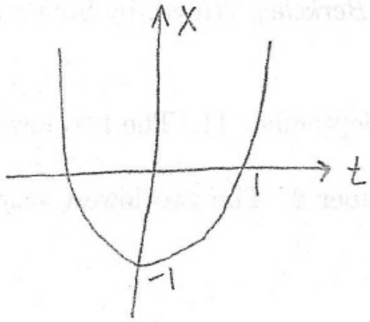
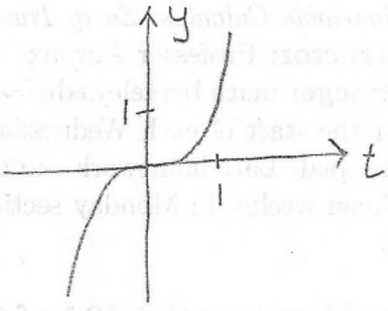
Practice problems §10.2: tangents to parametric curves

1) Use the graphs of $x = f(t)$ and $y = g(t)$ below to sketch the parametric curve $(x = f(t), y = g(t))$. Indicate with arrows the direction in which the curve is traced as t increases.





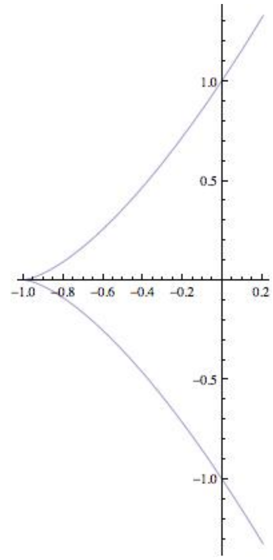
2) Make a guess for what $f(t)$ and $g(t)$ are above, and try to compute the slope of the tangent line to the parametric curve at $(-1, 0)$ by taking a limit of slopes of lines tending to the tangent line at $(-1, 0)$. Then verify your guess using the formula $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$.

3) Let $x = t\cos t$, $y = t\sin t$ for $-\pi \leq t \leq \pi$. a) Where is the self-intersection point of the curve? (Hint: see if you can find an axis of symmetry.) b) What are the equations of the two tangent lines at the self-intersection point?

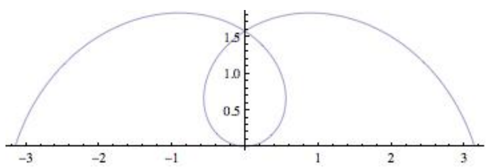
4) Recall the cycloid obtained by rolling a circle of radius r:

$$x = r(\theta - \sin\theta), \qquad y = r(1 - \cos\theta)$$

Find the slope of the tangent line in terms of θ. Where are the tangents horizontal and vertical?

Answers: 1) 

with direction going upwards. 2) $f(t) = t^2 - 1, g(t) = t^3$. The slope of the line from $(-1, 0)$ to a point (x, y) on the curve is $\frac{y-0}{x-(-1)} = \frac{y}{x+1} = \frac{t^3}{t^2} = t$ so as (x, y) approaches $(-1, 0)$, where $t = 0$, it seems like the slope should approach zero. This is indeed the case: $y'(t)/x'(t) = \frac{3}{2}t$ which equals zero when $t = 0$.

3) 

. The curve is symmetric over the y-axis because $x(t)$ is an odd function while $y(t)$ is even. It looks like two half-spirals, mirror images over the y-axis. The self-intersection point happens at $t = \pm\pi/2$, when $x = 0$. Equations of the two lines: find the two slopes $y'(t)/x'(t)$ for $t = \pm\pi/2$ to be $\mp 2/\pi$. So the lines are $y = \mp\frac{2}{\pi}x + \frac{\pi}{2}$.

4) $\frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta}$: horizontal tangents happen when this is zero, and vertical tangents happen when this becomes infinite. Numerator is zero when $\theta = \pi n$ for some integer n and the denominator is zero when $\theta = 2n\pi$. So they are both zero when $\theta = 2n\pi$. Using L'Hospital's rule we get $\frac{\cos\theta}{\sin\theta}$ which will go to infinity as $\theta \to 2n\pi$. Thus: horizontal slopes when $\theta = (2n+1)\pi$ and vertical slopes when $\theta = 2n\pi$. This can be seen in the picture for the cycloid.