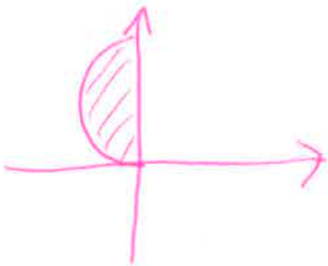
Practice problems: Double integrals in polar coordinates

1) Sketch the region whose area is given by the integral $\int_{\pi/2}^{\pi}\int_0^{2\sin\theta} r\,dr\,d\theta$ and evaluate the integral.



$$\int_{\pi/2}^{\pi} \frac{1}{2}(2\sin\theta)^2\,d\theta = \int_{\pi/2}^{\pi} 2\sin^2\theta\,d\theta$$

$$= \int_{\pi/2}^{\pi} (1-\cos(2\theta))\,d\theta$$

$$= \left[\theta - \frac{\sin(2\theta)}{2}\right]_{\pi/2}^{\pi} = \boxed{\pi/2}$$

2) By converting to polar coordinates, find $\int\int_R \sin(x^2+y^2)dA$, where R = region in the first quadrant between the circles with center the origin and radii 1 and 3.

$$\int_0^{\pi/2}\int_1^3 \sin(r^2)\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \left.\frac{-\cos(r^2)}{2}\right|_{r=1}^{3} d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}(-\cos(9)+\cos 1)\,d\theta$$

$$= \frac{\pi/2}{2}(\cos 1 - \cos 9) = \boxed{\frac{\pi}{4}(\cos 1 - \cos 9)}$$

3) Find the volume of the solid lying under the cone $z = \sqrt{x^2+y^2}$, above the xy-plane and inside the cylinder $x^2+y^2 = 2y$.

$$\int_0^{\pi}\int_0^{2\sin\theta} r\,(r\,dr\,d\theta) = \int_0^{\pi} \frac{1}{3}(2\sin\theta)^3\,d\theta$$

$$= \frac{8}{3}\int_0^{\pi}\sin^3\theta\,d\theta = \frac{8}{3}\int_0^{\pi}\sin\theta(1-\cos^2\theta)\,d\theta$$

$$= 8/3\left[\underbrace{-\cos\theta\Big|_0^{\pi}}_{2} - \underbrace{\int_0^{\pi}\cos^2\theta\sin\theta\,d\theta}_{2/3}\right]$$

$u = \cos\theta$
$du = -\sin\theta\,d\theta$
$\theta = 0 \Rightarrow u = 1$
$\theta = \pi \Rightarrow u = -1$

$$\int_1^{-1} u^2(-du) = \frac{1}{3}(2)$$

$\boxed{32/9}$

4) Find the volume of a sphere of radius a using polar coordinates.

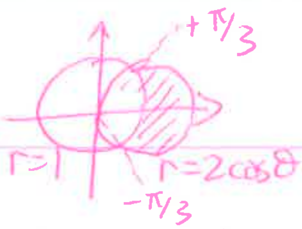
height at $(x,y)\in D$ is $2\sqrt{a^2-x^2-y^2}$

(recall sphere is $x^2+y^2+z^2=a^2$)

$$\int_0^{2\pi}\int_0^{a} 2\sqrt{a^2-r^2}\; r\,dr\,d\theta = (2\pi)\left.\frac{(a^2-r^2)^{3/2}}{-3/2}\right|_0^a$$

$$= (2\pi)\,\frac{2}{3}a^3 = \boxed{\frac{4}{3}\pi a^3}$$

5) Use a double integral to find the area of the region inside the circle $(x-1)^2+y^2 = 1$ and outside the circle $x^2+y^2 = 1$.



$1 = 2\cos\theta \Rightarrow \cos\theta = 1/2$

$\Rightarrow \theta = \pm\pi/3$

$$\iint_D dA = \int_{-\pi/3}^{\pi/3}\int_1^{2\cos\theta} r\,dr\,d\theta$$

$$= \int_{-\pi/3}^{\pi/3} \frac{1}{2}(4\underbrace{\cos^2\theta}_{\frac{1}{2}(1+\cos(2\theta))} - 1)\,d\theta$$

$$= \int_{-\pi/3}^{\pi/3} 1+\cos(2\theta) - \frac{1}{2}\,d\theta = \frac{1}{2}\left(\frac{2\pi}{3}\right) + \left.\frac{\sin 2\theta}{2}\right|_{-\pi/3}^{\pi/3}$$

$$= \boxed{\frac{\pi}{3} + \frac{\sqrt{3}}{2}}$$

Answers: 1) $\pi/2$, 2) $\frac{\pi}{4}(\cos 1 - \cos 9)$, 3) $32/9$, 4) $(4/3)\pi a^3$, 5) $\frac{\pi}{3} + \frac{\sqrt{3}}{2}$