Practice problems: Lagrange multipliers

1) Use Lagrange multipliers to find the maximum and minimum values of the function $f(x, y) = xy$ subject to the constraint $4x^2 + y^2 = 8$. $g(x,y) = 4x^2+y^2$

① $y = \lambda(8x)$ $\quad \lambda = 0 \Rightarrow x = y = 0$ but $g(0,0) \neq 8$

② $x = \lambda(2y)$ $\quad$ so $\lambda \neq 0$. Then $x \cdot ① = y \cdot ② \Rightarrow \cancel{\lambda} 8x^2 = \cancel{\lambda} 2y^2$ b/c $\lambda \neq 0$

$\Rightarrow y^2 = 4x^2 \Rightarrow y = \pm 2x$

Plug into constraint: $4x^2 + (2x)^2 = 8 \Rightarrow 8x^2 = 8 \Rightarrow x = \pm 1, y = \pm 2$

$$\boxed{\begin{array}{l} f(1,2) = f(-1,-2) = 2 \text{ max} \\ f(1,-2) = f(-1,2) = -2 \text{ min} \end{array}}$$

2) Find the points on the surface $z^2 = xy + 1$ closest to the origin using Lagrange multipliers.

$f(x,y,z) = x^2+y^2+z^2$, $g(x,y,z) = z^2 - xy - 1 = 0$

ⓘ $2x = -\lambda y$ $\quad \boxed{\text{EITHER}}$ ① $z \neq 0 \Rightarrow \lambda = 1$ $\boxed{\text{OR}}$ ② $z = 0 \Rightarrow xy = -1$

ⓘⓘ $2y = -\lambda x$

ⓘⓘⓘ $2z = \lambda 2z$

Case ① $\left.\begin{array}{l} 2x = -y \\ 2y = -x \end{array}\right\} \Rightarrow 2x = -y = \frac{1}{2}x \Rightarrow 4x = x \Rightarrow 3x = 0$

$\Rightarrow x = 0 = y$ b/c $2x = -y$

Then $z^2 = 1 \Rightarrow z = \pm 1$, $f(0,0,\pm 1) = 1$

Case ② $z = 0$, $xy = -1$. ⓘ $\cdot y =$ ⓘⓘ $\cdot x \Rightarrow -\lambda y^2 = -\lambda x^2$

$\lambda \neq 0$ (else $\lambda = 0 \Rightarrow x = 0$ but $xy = -1$ ※). So $x^2 = y^2 = (-1/y)^2 \Rightarrow y^4 = 1$

$\Rightarrow y = \pm 1 = -x$, $f(\pm 1, \mp 1, 0) = 2$ $\boxed{\text{So } (0,0,\pm 1) \text{ closest.}}$

b/c $xy = -1$

Note: (x,y,z) can get arb. large on $z^2 = xy + 1$ so there is no max.

3) Find the largest possible volume of a rectangular box, subject to its main diagonal having length L.

$f(x,y,z) = xyz$

$g(x,y,z) = x^2+y^2+z^2 = L^2$

$\nabla f = \lambda \nabla g \Rightarrow \begin{array}{ll} yz = 2\lambda x & \cdot x \\ xz = 2\lambda y & \cdot y \\ xy = 2\lambda z & \cdot z \end{array} \Rightarrow \lambda x^2 = \lambda y^2 = \lambda z^2$

$\lambda = 0 \Rightarrow$ y or $z = 0$ by $yz = 2\lambda x$ so then vol $= xyz$ is 0. ※

So $x^2 = y^2 = z^2$

$\Rightarrow x = y = z$ (≥ 0 b/c they are lengths)

So constraint $g(x,x,x) = 3x^2 = L^2$

$\Rightarrow x = L/\sqrt{3}$ & $\boxed{\text{max vol} = L^3/3\sqrt{3}}$

See next pg for solutions

4) Find the shortest distance from the point $(2, 0, -3)$ to the plane $x + y + z = 1$. *Problems 4 and 5 were on the previous worksheet – this time do them with Lagrange multipliers.*

5) A cardboard box without a lid is to have a volume of 32 cm^3. Find the dimensions that minimize the amount of cardboard used.

4) Minimize $f(x,y,z) = (x-2)^2 + y^2 + (z+3)^2$ subject to $g(x,y,z) = x+y+z = 1$.

$$\nabla f = \langle 2(x-2), 2y, 2(z+3) \rangle = \lambda \langle 1,1,1 \rangle = \lambda \nabla g$$

$$\Rightarrow \quad x-2 = y = z+3$$

$$\overset{\Rightarrow}{g=1} \quad x + (x-2) + (x-5) = 1$$

$$\Rightarrow 3x = 8$$

$$\Rightarrow x = 8/3$$

$$y = 2/3$$

$$z = -7/3$$

$$\Rightarrow \text{distance} = \sqrt{\left(\frac{8}{3}-2\right)^2 + \left(\frac{2}{3}\right)^2 + \left(-\frac{7}{3}+3\right)^2} = \boxed{\frac{2}{\sqrt{3}} \text{ as before}}$$

5) Minimize $f(w,h,l) = 2wh + 2lh + wl$ subject to $g(w,h,l) = whl = 32$.

$$\nabla f = \lambda \nabla g \Rightarrow \left.\begin{matrix} (2h + l = \lambda h l) \cdot w \\ (2w + 2l = \lambda w l) \cdot h \\ (2h + w = \lambda w h) \cdot l \end{matrix}\right\} \Rightarrow \begin{matrix} 2\cancel{h}w + lw \overset{①}{=} 2\cancel{h}w + 2lh \\ 2hw + \cancel{2lh} \overset{②}{=} 2hl + lw \end{matrix}$$

① $\Rightarrow l(w-2h) = 0$

② $\Rightarrow w(l-2h) = 0$

$whl = 32 \Rightarrow l, w, h$ are $\neq 0$

so ① $\Rightarrow w = 2h$

② $\Rightarrow l = 2h$

then $g = 32 \Rightarrow (2h) \cdot h \cdot (2h) = 4h^3 = 32 \Rightarrow h^3 = 8$

$\Rightarrow h = 2$

so $w = 2h = 4$, $l = 2h = 4 \Rightarrow \boxed{(w,h,l) = (4,2,4)}$ as we found