Practice problems: Gradient, directional derivative

1) Find the directional derivative of $f(x,y) = e^x \sin y$ at the point $(0, \pi/3)$ in the direction $\vec{v} = \langle -6, 8 \rangle$. $\nabla f = \langle f_x, f_y \rangle = \langle e^x \sin y, e^x \cos y \rangle$

$\Rightarrow \nabla f|_{(0,\pi/3)} = \langle \sqrt{3}/2, \frac{1}{2} \rangle$

$\frac{\vec{v}}{|\vec{v}|} = \langle -3/5, 4/5 \rangle \Rightarrow D_{\vec{v}} f = \nabla f \cdot \frac{\vec{v}}{|\vec{v}|} = \frac{1}{10} \langle \sqrt{3}, 1 \rangle \cdot \langle -3, 4 \rangle$

$= \boxed{\frac{1}{10}(-3\sqrt{3} + 4)}$

2) Recall the problem on last worksheet involving an electric potential of $V(x,y,z) = 5x^2 - 3xy + xyz$ in a region of space. a) In which direction does V change most rapidly at P? (b) What is the maximum rate of change at P? (You don't need to simplify your expression in (b).) $P = P(0, 4/5, -3/5)$

a) $\nabla V|_P = \boxed{\langle 38, 6, 12 \rangle}$

b) $|\nabla V|_P| = \sqrt{38^2 + 6^2 + 12^2}$

3) Find the equation of the tangent plane to the surface $x + y + z = e^{xyz}$ at the point $(0, 0, 1)$.

Let $F(x,y,z) = x + y + z - e^{xyz}$. Then surface is level set of F, where $F = 0$, so ∇F will be $\perp$ to this surface.

$\nabla F = \langle 1 - yze^{xyz}, 1 - xze^{xyz}, 1 - xye^{xyz} \rangle$

$\nabla F|_{(0,0,1)} = \langle 1, 1, 1 \rangle$ = normal $\vec{n}$, $\vec{r_0} = \langle 0, 0, 1 \rangle$ endpoint on the plane

$\Rightarrow$ equation is $\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0 \Rightarrow 1(x-0) + 1(y-0) + 1(z-1) = 0$

$\Rightarrow \boxed{x + y + z = 1}$

4) Are there any points on the hyperboloid $x^2 - y^2 - z^2 = 1$ where the tangent plane is parallel to the plane $z = x + y$? i.e. tangent plane to hyperboloid has normal parallel to that of $x+y-z=0$, which is $(1,1,-1)$.

hyperboloid: level surface of $F(x,y,z) = x^2 - y^2 - z^2$

$\nabla F = \langle 2x, -2y, -2z \rangle$. Does there exist (x,y,z) so $\nabla F = \lambda(1,1,-1)$ for some λ? Then

$2x = \lambda$
$-2y = \lambda$ $\Rightarrow$ $x = -y = z$. But then $x^2 - y^2 - z^2 = x^2 - x^2 - x^2 = -x^2$
$-2z = -\lambda$ and $-x^2 = 1$ is a contradiction, So $\boxed{\text{no.}}$

5) Suppose that the directional derivatives of $f(x, y)$ are known at a given point in two non-parallel directions given by unit vectors $\hat{u}$ and $\hat{v}$. Is it possible to find ∇f at this point? If so, how would you do it? This uses a little linear algebra.

$$\nabla f \cdot \hat{u} = f_x u_1 + f_y u_2 = a$$
$$\nabla f \cdot \hat{v} = \underbrace{f_x v_1 + f_y v_2}_{\text{system of 2 linear equations}} = b$$

system of 2 linear equations $\Rightarrow$ can use matrices

$$\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \begin{pmatrix} f_x \\ f_y \end{pmatrix} = \begin{pmatrix} f_x u_1 + f_y u_2 \\ f_x v_1 + f_y v_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

b/c $\hat{u}, \hat{v}$ are linearly independent; $\det \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \neq 0$ and the matrix has an inverse. So we can find ∇f explicitly as

$$\begin{pmatrix} f_x \\ f_y \end{pmatrix} = \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \end{pmatrix}$$

so answer is $\boxed{\text{yes}}$.