Practice problems: more chain rule, gradient

1) Use a tree diagram to write out the Chain Rule for $u = f(x,y)$ and $x = x(r,s,t)$, $y = y(r,s,t)$.

2) Let $w = xy + yz + zx$, $x = r\cos\theta$, $y = r\sin\theta$, $z = r\theta$. Use the Chain Rule to find $\partial w/\partial r$ and $\partial w/\partial\theta$ at $(r,\theta) = (2, \pi/2)$. $x(2,\pi/2) = 0,\ y(2,\pi/2) = 2,\ z(2,\pi/2) = \pi$

- $\dfrac{\partial w}{\partial r} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial r} + \dfrac{\partial w}{\partial z}\dfrac{\partial z}{\partial r} = (y+z)(\cos\theta) + (x+z)(\sin\theta) + (y+x)(\theta)$

$= (2+\pi)(0) + (0+\pi)(1) + (2)(\pi/2) = \boxed{2\pi}$

- $\partial w/\partial\theta = (y+z)(-r\sin\theta) + (x+z)(r\cos\theta) + (y+x)(r)$

$= (2+\pi)(-2) + 0 + (2)(2) = \boxed{-2\pi}$

3) The temperature at a point (x,y) is $T(x,y)$, measured in degrees Celsius. A bug crawls so that its position after t seconds is given by $x = \sqrt{1+t}$, $y = 2 + \frac{1}{3}t$, where x and y are measured in centimeters. The temperature function satisfies $T_x(2,3) = 4$ and $T_y(2,3) = 3$. How fast is the temperature rising on the bug's path after 3 seconds?

$x(3) = 2,\ y(3) = 3$

$$\left.\frac{dT}{dt}\right|_{t=3} = \left.\frac{\partial T}{\partial x}\right|_{(2,3)} \left.\frac{dx}{dt}\right|_{t=3} + \left.\frac{\partial T}{\partial y}\right|_{(2,3)} \left.\frac{dy}{dt}\right|_{t=3}$$

$$= 4\left(\frac{1}{2\sqrt{1+3}}\right) + 3\left(\frac{1}{3}\right)$$

$$= \boxed{2^\circ C/s}$$

4) Find ∇f for $f(x,y,z) = x^2yz - xyz^3$. Evaluate the gradient at the point $P(2,-1,1)$. Find the rate of change of f at P in the direction of the vector $\vec{u} = \langle 0, 4/5, -3/5 \rangle$.

- $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle = \langle 2xyz - yz^3, x^2z - xz^3, x^2y - 3xyz^2 \rangle$
- $\nabla f|_{(2,-1,1)} = \langle -4+1, 4-2, -4+6 \rangle = \boxed{\langle -3, 2, 2 \rangle}$
- $\vec{u}$ unit so $D_{\vec{u}} f = \nabla f|_{(2,-1,1)} \cdot \vec{u} = \langle -3, 2, 2 \rangle \cdot \langle 0, 4/5, -3/5 \rangle = \boxed{2/5}$

5) Show that $\nabla(uv) = u\nabla v + v\nabla u$ for u, v differentiable functions of x and y. and z.

$\nabla(uv) = \langle \frac{\partial}{\partial x}(uv), \frac{\partial}{\partial y}(uv), \frac{\partial}{\partial z}(uv) \rangle$

$= \langle u_x v + u v_x, u_y v + u v_y, u_z v + u v_z \rangle = v \langle u_x, u_y, u_z \rangle + u \langle v_x, v_y, v_z \rangle$

$= \boxed{u \nabla v + v \nabla u}$

6) Suppose that over a certain region of space the electric potential is given by $V(x,y,z) = 5x^2 - 3xy + xyz$. Find the rate of change of the potential at $P(3,4,5)$ in the direction of the vector $\vec{v} = \frac{1}{\sqrt{3}}\left(\hat{\imath} + \hat{\jmath} - \hat{k}\right)$. $\vec{v}$ unit so want $\nabla V|_P \cdot \frac{1}{\sqrt{3}} \langle 1, 1, -1 \rangle$

$\nabla V = \langle 10x - 3y + yz, -3x + xz, xy \rangle$

$\nabla V|_{(3,4,5)} = \langle 38, 6, 12 \rangle$

$\Rightarrow D_{\vec{u}} V = \langle 38, 6, 12 \rangle \cdot \frac{1}{\sqrt{3}} \langle 1, 1, -1 \rangle$

$= \boxed{\frac{32}{\sqrt{3}}}$