Practice problems: linear approximation, implicit differentiation, multivariable chain rule

1) Find the linearization of the function $f(x,y) = x^2e^y$ at the point $(1,0)$. Then estimate $f(1.1, 0.02)$.

$\frac{\partial f}{\partial x} = 2xe^y \Rightarrow f_x(1,0) = 2$

$\frac{\partial f}{\partial y} = x^2e^y \Rightarrow f_y(1,0) = 1$

linearization: $f(1,0) = 1 = z_0$ & $(x_0, y_0) = (1,0)$

$\Rightarrow \boxed{z - 1 = 2(x-1) + 1(y-0)}$

$f(1.1, 0.02)$

$\approx 1 + 2(0.1) + 1(.02)$

$= \boxed{1.22}$

2) Find $\frac{dz}{dt}$ where $z = \sqrt{1+x^2+y^2}$ and $x = \ln t$, $y = \cos t$.

$z_x = \frac{x}{z}$, $z_y = \frac{y}{z}$, $x' = \frac{1}{t}$, $y' = -\sin t$

$\Rightarrow \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \boxed{\frac{x}{z}\cdot\frac{1}{t} + \frac{y}{z}(-\sin t)}$

3) Use implicit differentiation to find $\partial z/\partial x$ and $\partial z/\partial y$ where $e^z = xyz$.

- $\frac{\partial}{\partial x}(e^z = xyz) \Rightarrow e^z z_x = yz + xyz_x$

  $\Rightarrow z_x(e^z - xy) = yz$

  $\Rightarrow \boxed{z_x = \frac{yz}{e^z - xy}}$

- let $F(x,y,z) = e^z - xyz$. $\quad 0 = \frac{\partial F}{\partial x}\overbrace{\frac{\partial x}{\partial y}}^{0} + \frac{\partial F}{\partial y}\overbrace{\frac{\partial y}{\partial y}}^{1} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y}$

$\Rightarrow \frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \boxed{\frac{xz}{e^z - xy} = z_y}$

Makes sense by symmetry.

c) ctd

$\Rightarrow 3\frac{dL}{dt} = 1\cdot 2 + 2\cdot 2 + 2\cdot(-3) = 0 \Rightarrow \boxed{\frac{dL}{dt} = 0\, m/s}$

4) The length ℓ, width w and height h of a box change with time. At a certain instant the dimensions are $\ell = 1$ m and $w = h = 2$ m, and ℓ and w are increasing at a rate of 2 m/s while h is decreasing at a rate of 3 m/s. At that instant find the rates at which the following quantities are changing: (a) the volume, (b) the surface area and (c) the length of the main box diagonal.

a) $V(\ell, w, h) = \ell w h$, $V_\ell = wh$ $V_h = \ell w$ $V_w = \ell h$

t_0 = instant

$$\left.\frac{dV}{dt}\right|_{t_0} = V_\ell \cdot \ell' + V_h \cdot h' + V_w \cdot w' = (2\cdot 2)(2) + (1\cdot 2)(-3) + (1\cdot 2)(2) = \boxed{6\, m^3/s}$$

b) $S(\ell, w, h) = 2(\ell w + \ell h + wh)$, $S_\ell = 2(w+h)$, $S_w = 2(\ell + h)$, $S_h = 2(\ell + w)$

$$\left.\frac{dS}{dt}\right|_{t=t_0} = S_\ell \cdot \ell' + S_w \cdot w' + S_h \cdot h' = [2(2+2)](2) + 2(1+2)(2) + 2(1+2)(-3)$$
$$= 16 + 12 - 18 = \boxed{10\, m^2/s}$$

c) $L = \sqrt{\ell^2 + w^2 + h^2} \Rightarrow L^2 = \ell^2 + w^2 + h^2 \Rightarrow 2L\frac{dL}{dt} = 2\ell\frac{d\ell}{dt} + 2w\frac{dw}{dt} + 2h\frac{dh}{dt}$ $\Rightarrow$ ctd at top

$L(t_0) = \sqrt{1+4+4} = 3$

5) The radius of a right circular cone is increasing at a rate of 1.8 in/s while its height is decreasing at a rate of 2.5 in/s. At what rate is the volume of the cone changing when the radius is 120 in and the height is 140 in?

Volume $V = \frac{1}{3}(\text{base})(\text{height}) = \frac{1}{3}\pi r^2 \cdot h$.

$$\frac{\partial V}{\partial r} = \frac{2}{3}\pi r h, \quad \frac{\partial V}{\partial h} = \frac{1}{3}\pi r^2, \quad \frac{dr}{dt} = 1.8, \quad \frac{dh}{dt} = -2.5$$

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt}$$

$$= \left(\tfrac{2}{3}\pi \cdot 120 \cdot 140\right)(1.8) + \left(\tfrac{1}{3}\pi \cdot 120^2\right)(-2.5)$$

$$= 20160\pi - 12000\pi$$

$$= 8160\pi \approx \boxed{25635\, in^3/s}$$