Practice Problems: Contour plots

1) Sketch contour plots and graphs of the following functions: a) $z = \sin(xy)$, b) $z = e^x \cos y$, c) $z = \sin(x-y)$, d) $z = \sin x - \sin y$, e) $z = (1-x^2)(1-y^2)$, f) $z = \frac{x-y}{1+x^2+y^2}$. If you find this hard, then see page 902 in the book, exercises 61 – 66 of §14.1, for which you need to match these functions with given contour maps and graphs.

Answers to b) and e): Some of these are HW problems so I will explain solutions for two.

b) $z = e^x \cos y$: if we fix $y = c$ (i.e. slice the surface with the plane placed at $y = c$) then the curve we see is $z = e^x \cos c$. This is an exponential function in x, so the height increases exponentially as x increases, and goes to 0 as x goes to $-\infty$. On the other hand, if we fix a value of $x = c$, then the section of the surface will look like a cosine function with amplitude e^c. Larger c means bigger amplitude, smaller c (x closer to 0) means smaller amplitude. This matches with the first surface in (A). To find the contour plot: pick a point on the surface, say $(1, \pi/4, f(1, \pi/4)) = (1, \pi/4, e \cdot \frac{\sqrt{2}}{2})$. We want to figure out, in what direction do we walk on the surface so that we stay at constant height? If we walk only in the increasing x direction, we are walking upwards along an exponential function, which is no good. Similarly if we walk in the decreasing x direction we go down. So we go around a curve, walking around the "bump" in the picture of (A). In other words, we walk along the contours drawn in IV.

e) $z = (1-x^2)(1-y^2)$: for this one we can consider the behavior as x and y get large. Regardless of whether they are going to $+\infty$ or $-\infty$, squaring them makes both go to $+\infty$. So we have a product of $1-x^2$ and $1-y^2$, each of which go to $-\infty$, but their product will go to $+\infty$. In particular, along the lines $x = \pm y$, the curve is growing like the 4th order polynomial $(1-x^2)(1-x^2)$. So we should have something that has four upturned corners, because the height grows in the four directions $x = \pm y$. This matches with B. Another feature is, when $x = 0$ we get a downward facing parabola $1 - y^2$, which can be seen in the picture (although it is fairly flat-looking). As x increases we go through height $z = 0$ until we get to $x > 1$ and then we get an upward facing parabola $c'(1-y^2)$ in the plane $x = c > 1$ because $c' = 1 - c^2 < 0$ will make the coefficient on y^2 positive. Once we have the graph B, we can see that walking in box-like circles near the origin keeps us at the same height, and walking around the four upturned corners also keeps us at the same height, so the contours are as in VI.

So here are some ways we can picture a graph: look at level sets $f(x, y) = c$ (or in three variables we get level surfaces $f(x, y, z) = c$)), look at the behavior of f as x and y get large, see how the height $z = f(x, y)$ varies as we fix one coordinate and vary the other, and look at bounds on the height (for example the trig functions sine and cosine are always between -1 and 1). Another tip: if the function $f(x, y)$ is a function of $\sqrt{x^2 + y^2}$, say $f(x, y) = g(\sqrt{x^2 + y^2}) = g(r)$ (so in the case of $f(x, y) = 3 - x^2 - y^2$ we can take $g(r) = 3 - r^2$), then instead of plotting level sets $f(x, y) = c$ we can plot level sets where the constant is $g(r)$ for fixed r. Then circles of radius r are level sets, and they map to a height of $g(r)$ in the graph.