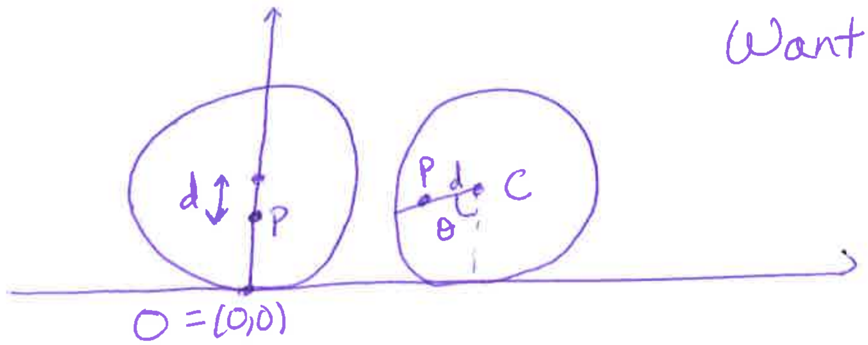
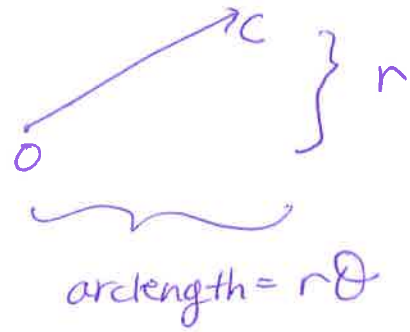
Review problems

P1)

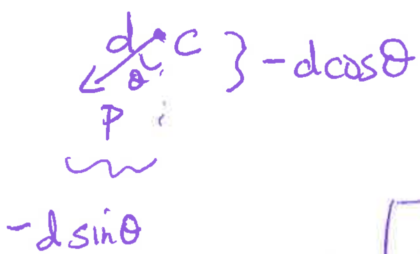


Want $\vec{OP}$.



$$\vec{OP} = \vec{OC} + \vec{CP}$$

$$\vec{OC} = \langle r\theta, r \rangle$$

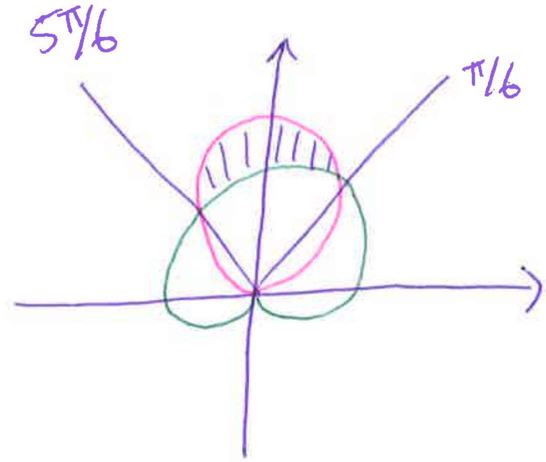


$$\vec{CP} = \langle -d\sin\theta, -d\cos\theta \rangle$$

$$\Rightarrow \boxed{\vec{OP} = \langle r\theta - d\sin\theta,\ r - d\cos\theta \rangle}$$

P2) $r = 3\sin\theta$

$r = 1 + \sin\theta$



$$3\sin\theta = 1 + \sin\theta \Rightarrow \sin\theta = \frac{1}{2}$$

$$\Rightarrow \theta = \pi/6,\ 5\pi/6$$

$$\text{area} = \int_{\frac{\pi}{6}}^{5\pi/6} \int_{1+\sin\theta}^{3\sin\theta} r\, dr\, d\theta = \pi$$

P3) plane we want is $\perp$ to 2 planes

$\Leftrightarrow$ normal we want is $\underbrace{\perp \text{ to }}_{\text{so take their cross product}}$ 2 normals

$$\begin{vmatrix} i & j & k \\ 2 & 1 & -2 \\ 1 & 0 & 3 \end{vmatrix} = \langle 3, -8, -1 \rangle$$

$$\langle 3, -8, -1 \rangle \cdot \langle x-1, y-5, z-1 \rangle = 0$$

$$\Leftrightarrow \boxed{3x - 8y - z = -38}$$

P4) $u' = \left(\vec{r} \cdot (\vec{r}\,' \times \vec{r}\,'')\right)' = \vec{r}\,' \cdot \overbrace{(\vec{r}\,' \times \vec{r}\,'')}^{0 \text{ b/c } \vec{r}\,' \times \vec{r}\,'' \perp \vec{r}\,'} + \vec{r} \cdot (\vec{r}\,' \times \vec{r}\,'')'$

$= \vec{r} \cdot (\underbrace{\vec{r}\,'' \times \vec{r}\,''}_{0}) + \vec{r} \cdot (\vec{r}\,' \times \vec{r}\,''')$

used product rule twice in differentiation

$= \boxed{\vec{r} \cdot (\vec{r}\,' \times \vec{r}\,''')}$

P5) $f(x,y) = \sqrt{y + \cos^2 x} \approx f(0,0) + \nabla f\big|_{(0,0)} \cdot \langle x-0, y-0 \rangle$

$= 1 + \underbrace{\left\langle \frac{-\sin x \cos x}{\sqrt{y + \cos^2 x}}, \frac{1}{2\sqrt{\cos^2 x + y}} \right\rangle \Big|_{(0,0)}}_{\langle 0, \frac{1}{2} \rangle} \cdot \langle x, y \rangle$

$= \boxed{1 + \frac{y}{2}}$

P6) $D_{\hat{u}} f = \nabla f\big|_{(1,1)} \cdot \frac{\hat{u}}{|\hat{u}|} = \langle 4x, 3 \rangle\big|_{(1,1)} \cdot \frac{1}{\sqrt{5}} \langle 1, 2 \rangle$

$= \langle 4, 3 \rangle \cdot \frac{1}{\sqrt{5}} \langle 1, 2 \rangle = \frac{10}{\sqrt{5}} = \boxed{2\sqrt{5}}$

P7) $\nabla f = \langle 3x^2+3y, 3y^2+3x \rangle = \langle 0,0 \rangle$

$\Rightarrow \left.\begin{matrix} x^2=-y \\ y^2=-x \end{matrix}\right\} \Rightarrow x^2 = y^4 = -y \Rightarrow y^4+y=0$

$\Rightarrow y(y^3+1)=0$

crit pts $= (0,0)$ & $(-1,-1)$

$\Rightarrow y=0 \therefore x=0$

$y=-1 \therefore x=-1$

nature? $D = \det\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \det\begin{pmatrix} 6x & 3 \\ 3 & 6y \end{pmatrix} = 36xy-9$

$D(0,0) = -9 < 0 \Rightarrow \boxed{(0,0) \text{ saddle}}$

$D(-1,-1) = 36-9>0$

$f_{xx} = -6 < 0 \Rightarrow \boxed{(-1,-1) \text{ local max}}$

Global behavior? Domain $=\mathbb{R}^2$.

fix $x=x_0$, $y \to +\infty$, $f(x_0,y) = x_0^3+y^3+3x_0y-27 \to +\infty$

b/c y^3 dominates

fix $x=x_0$, $y\to -\infty$ then $f\to -\infty$

$y=y_0$, $x\to +\infty \Rightarrow f\to +\infty$

$y=y_0$ $x\to -\infty \Rightarrow f \to -\infty$

Since $f\to +\infty$ and $-\infty$ on bdry at infinity,
no global max/min.

P8) $\frac{\partial g}{\partial s} = \frac{\partial f}{\partial s} = f_x \cdot \frac{\partial x}{\partial s} + f_y \frac{\partial y}{\partial s} = 2s\cdot(f_x - f_y)$ $\quad x=s^2-t^2, \; y=t^2-s^2$

$\frac{\partial g}{\partial t} = 2t(f_y - f_x)$ $\quad \therefore t\cdot\frac{\partial g}{\partial s} + s\cdot\frac{\partial g}{\partial t} = 2st(f_x-f_y+f_y-f_x)$

$=0$

P9)

- new bounds
- Jacobian
- new integrand

new bounds

boundary equation: $x^2 - xy + y^2 = 2$

Convert: $\left(\sqrt{2}u - \sqrt{\tfrac{2}{3}}v\right)^2 - \left(\sqrt{2}u - \sqrt{\tfrac{2}{3}}v\right)\left(\sqrt{2}u + \sqrt{\tfrac{2}{3}}v\right) + \left(\sqrt{2}u + \sqrt{\tfrac{2}{3}}v\right)^2 = 2$

$\Rightarrow 2u^2 + 2v^2 = 2 \Rightarrow u^2 + v^2 = 1$

So new domain has boundary $u^2+v^2=1$ hence new domain is $u^2+v^2 \leq 1$.

Jacobian

$$dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$= \left|\det\begin{pmatrix}\sqrt{2} & -\sqrt{\tfrac{2}{3}} \\ \sqrt{2} & \sqrt{\tfrac{2}{3}}\end{pmatrix}\right| du\,dv = \frac{4}{\sqrt{3}}\,du\,dv$$

new integrand

$x^2 - xy + y^2 + 5$ converts to

$2u^2 + 2v^2 + 5$ in u,v coordinates

Putting together

$$\iint_R x^2 - xy + y^2 + 5\,dA = \iint_{u^2+v^2 \leq 1} \left(2u^2+2v^2+5\right)\frac{4}{\sqrt{3}}\,du\,dv$$

$$= \int_0^{2\pi}\int_0^1 \left(2r^2+5\right)\frac{4}{\sqrt{3}}\,r\,dr\,d\theta = \boxed{\frac{24\pi}{\sqrt{3}}}$$

P10) a) $\text{mass} = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\phi} k \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \dfrac{k\pi}{8}$

c.o.m: by symmetry of region & density
$\bar{x} = \bar{y} = 0$

$$\bar{z} = \frac{1}{\text{mass}} \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\phi} k \cdot \overbrace{(\rho\cos\phi)}^{z} \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

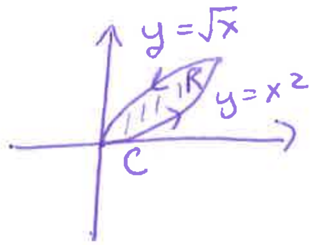
$$= \frac{7\cancel{\pi}/96}{\cancel{\pi}/8} = \frac{7}{12} \Rightarrow \boxed{\text{c.o.m.} = \left(0, 0, \tfrac{7}{12}\right)}$$

b) $(\text{dist to } z\text{-axis})^2 = x^2 + y^2 = (\rho\sin\phi)^2$

$$I_z = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\phi} k \cdot (\rho\sin\phi)^2 \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \boxed{\frac{k \cdot 11\pi}{960}}$$

P11)



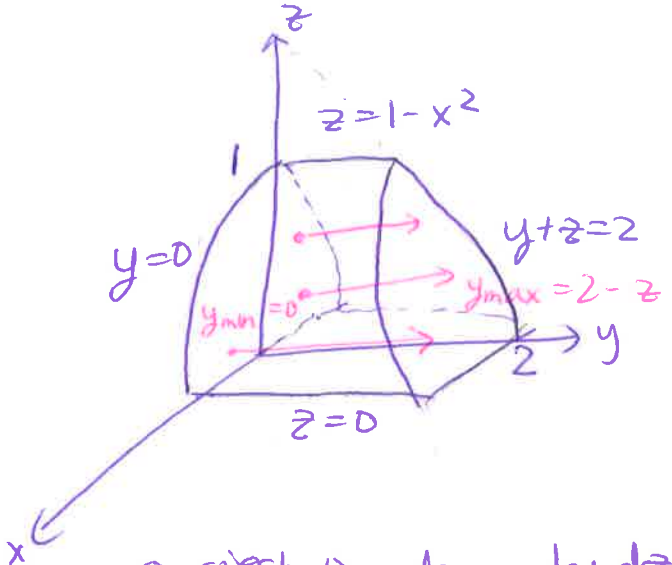
$\boxed{\text{out of curve}}$ $\text{Flux} = \oint_C \vec{F} \cdot \underbrace{\hat{n}\,ds}_{\langle dy, -dx \rangle}$

$$\xrightarrow{\text{2D div thm}} = \iint_R \text{div}\,\vec{F} \, dA$$

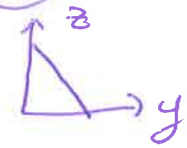
$$= \int_0^1 \int_{x^2}^{\sqrt{x}} 2 - 1 \, dA$$

$$= \int_0^1 \sqrt{x} - x^2 \, dx = \boxed{\frac{1}{3}}$$

P12) $\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \, dV$ b/c S is closed



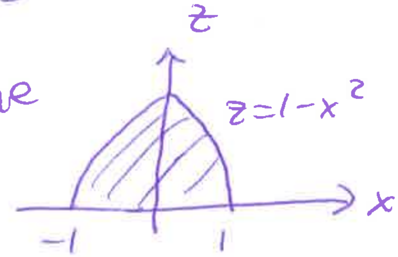
Slice taco-shape $z=1-x^2$ with plane $y+z=2$



easiest to do $dy\,dz\,dx$

(if e.g. $dz\,dy\,dx$ then splits into 2 integrals where either $z=1-x^2$ or $z=2-y$ is upper bound)

$\underline{dy\,dz\,dx}$ Shadow on (x,z) plane



$-1 \leq x \leq 1$

$0 \leq z \leq 1-x^2$

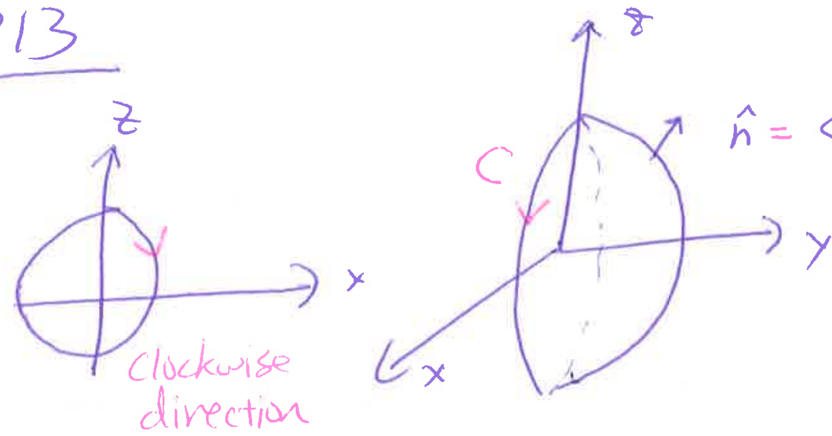
At each (x,z), $0 \leq y \leq 2-z$

$\operatorname{div} \vec{F} = 3y$ so $\iint_S \vec{F} \cdot d\vec{S} = \int_{-1}^{1} \int_0^{1-x^2} \int_0^{2-z} 3y \, dy\,dz\,dx$

$= \boxed{\dfrac{184}{35}}$

don't plug in anything for y, only may need to do that in line & surface integrals

P13



$\hat{n} = \langle x, y, z \rangle$

C: $\vec{r}(t) = \langle -\cos t, 0, \sin t \rangle$ $\quad 0 \le t \le 2\pi$

so moves clockwise

$\text{curl}\,\vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ y & z & x \end{vmatrix} = \langle -1, -1, -1 \rangle$

$$\iint_S \text{curl}\,\vec{F} \cdot d\vec{S} = \iint_S \langle -1,-1,-1 \rangle \cdot \langle x, y, z \rangle \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{\pi}\int_0^{\pi} \sin\phi \left(\underbrace{-\sin\phi\cos\theta}_{\to 0} - \sin\phi\sin\theta - \cos\phi \right) d\phi \, d\theta$$

$$= -\pi \qquad \text{b/c } \int_0^{\pi} \cos\theta \, d\theta = \sin\theta \Big|_0^{\pi} = 0$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle 0, \sin t, -\cos t \rangle \cdot \langle \sin t, 0, \cos t \rangle \, dt$$

$$= \int_0^{2\pi} -\cos^2 t \, dt = \boxed{-\pi}$$