Math 53 Discussion: Review for Midterm 2

Lagrange multipliers. §14.8

- Lagrange multipliers: to find the extreme values of some function f subject to a condition $g = c$ for some constant c, solve for $\nabla f = \lambda \nabla g$ and $g = c$, a system of three equations in three unknowns in 2D, or four equations and four unknowns in 3D. If the constraining curve/surface is bounded and smooth (i.e. no corners), then a maximum and minimum of f is always achieved, and can be checked by plugging the points into f. If the domain is unbounded, then we need to check the behavior of f as we tend to infinity.

Double integrals. §15.1 – 15.6

- Cartesian and polar coordinates. §15.1–15.4.
  - Analogous to an integral in 1D being the area under a curve, we can think of integrals in 2D as the volume under a surface. Integration can be done in the order $dx\,dy$ or $dy\,dx$. E.g. if the integral is in the order $dx\,dy$ over a region R, take the shadow of R on the y-axis to get the y-bounds, then see how x varies at each such y.
  - In polars, the area element is $dA = r\,dr\,d\theta$.
- Applications – mass, center of mass, average value of a function. §15.5
  - Mass of lamina (infinitely thin sheet of paper) occupying 2D region R, with density ρ: integrate density over the region R to get mass equals $m = \int\int_R \rho\,dA$.
  - Center of mass $(\overline{x}, \overline{y})$ is obtained by finding the average x and y coordinates, weighted by the density.

$$\overline{x} = \frac{1}{m} \int\int_R x\rho\,dA$$

    $\overline{y}$ is similar.
  - Average value of a function f over region R is $\frac{1}{\text{area}(R)} \int\int_R f\,dA$.
- Surface area. §15.6
  - Surface area of a surface obtained as a graph $z = f(x, y)$ over region R in the xy-plane is

$$\int\int_R \sqrt{{f_x}^2 + {f_y}^2 + 1}\,dx\,dy$$

Triple integrals. §15.7–15.9

- Triple integrals involve integrating a function f over a 3D solid. We can think of this as finding the mass of such a solid with density f.
- Method: e.g. suppose the area element is $dx\,dy\,dz$, and we want to integrate over a solid E. First find the shadow of E on the yz-plane (this will be the slice parallel to the yz-plane where E is "widest" and not necessarily the same as setting $x = 0$ in the equation for E). Express the region for this shadow as you would a double integral to get the y and z bounds. Then at each point (y, z) in this region, see how x varies as a function of y and z.
- Cylindrical coordinates (r, θ, z): these are 2D polar coordinates along with the third coordinate being z. The volume element is $dV = r\,dr\,d\theta\,dz$.
- Spherical coordinates (ρ, θ, ϕ): here ρ is the distance of the point to the origin, θ is the same as in cylindrical, i.e. the angle from the positive x-axis when projected to the xy-plane, and ϕ is the angle from the positive z-axis. Note that ϕ takes values between 0 and π.
- Applications: analogous to the formulas above. Now density is a function of three variables, and for average value of a function we divide by volume instead of area.

Change of variables using the Jacobian. §15.10

- Allows us to evaluate integrals by changing variables so the domain is nicer. Examples include polar coordinates in 2D, and spherical and cylindrical coordinates in 3D.
- Method in 2D (3D is similar):
  1. Change from region R in xy-plane to region S in uv-plane by $x = x(u,v), y = y(u,v)$. E.g. see where the boundaries of R map to by finding equations describing them in x and y, then plugging in $x(u,v)$ and $y(u,v)$ to find the corresponding equations in u and v of the boundaries for S.
  2. The area element dA becomes $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv$. (One way to remember what goes on top and on bottom is to think of $\partial(u,v)$ as "canceling" $du\, dv$ to give $dx\, dy$. Mathematically this is not what's happening, but it's a way to remember the order.)
  3. Here $\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$ is called the <u>Jacobian</u> of the transformation. So the first order partials of x go in the first row and of y go in the second row. (We have a 3×3 matrix in 3D.) We take the <u>absolute value</u> of this determinant when evaluating the integral.
  4. Evaluate the double integral now in terms of u and v.
- If we're given u and v in terms of x and y there are two ways to find $\frac{\partial(x,y)}{\partial(u,v)}$:
  1. Solve for x and y as functions of u and v and find the partial derivatives directly.
  2. Find $\frac{\partial(u,v)}{\partial(x,y)}$ and then take the reciprocal to get what we want, i.e. $\frac{\partial(x,y)}{\partial(u,v)}$. In other words,

$$\frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{\partial(x,y)}{\partial(u,v)}$$

- To summarize:
  1. Find the new region S in the uv-plane and the new bounds describing S.
  2. The area element becomes $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv$.
  3. Write the function we're integrating, $f(x,y)$, in terms of u and v as $f(x(u,v), y(u,v))$.
  4. Evaluate the integral now written in terms of u and v, using the usual tools for multiple integrals.