Math 53 Discussion: Review for Midterm 1

Curves: Sections 10.1–10.4

- Parametric equations: $x = f(t)$, $y = g(t)$ for some functions f, g.
  Self-intersections of curves occur when two different values of t give same point (x, y).
  Derivative dy/dx at a point on the curve can be computed via $dy/dx = \frac{dy/dt}{dx/dt}$.

- Length of curves: length of small piece of curve is $\Delta s \approx \sqrt{\Delta x^2 + \Delta y^2}$ so length L of curve traced out from time t_1 to time t_2 is $L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$.

- Area under a curve: $A = \int_{x_1}^{x_2} y dx = \int_{t_1}^{t_2} y(t) \frac{dx}{dt} dt$. Want $x_1 < x_2$ so we're integrating from left to right, however it may be that t_1 is greater than t_2.

- Polar coordinates and curves: (r, θ) where r is the distance from the origin, θ is the angle from the positive x-axis. $x = r\cos\theta$, $y = r\sin\theta$, $x^2 + y^2 = r^2$ and $\tan\theta = y/x$.
  A polar curve is of the form $r = f(\theta)$.

- Polar areas: find the range of the angle θ in the area you are trying to compute, then $A = \int_{\theta_1}^{\theta_2} \frac{1}{2} r(\theta)^2 d\theta$. Sometimes θ_1, θ_2 were the intersection points of two polar curves, and we found the area between the two curves.

Vectors: Sections 12.1–12.4

- Vectors: how to manipulate vectors, such as adding and subtracting

- Dot product: This gives us a scalar. There are two ways of writing the dot product $\overrightarrow{a} \cdot \overrightarrow{b} = \sum_i a_i b_i$ and $|\overrightarrow{a}|\,|\overrightarrow{b}|\cos\theta$ where θ is the angle (at most π) between $\overrightarrow{a}$ and $\overrightarrow{b}$.
  Scalar and vector projections: projecting a vector $\overrightarrow{a}$ onto $\overrightarrow{b}$ means finding the component of $\overrightarrow{a}$ in the (unit) direction of $\overrightarrow{b}$. That is, $\text{comp}_{\overrightarrow{b}} \overrightarrow{a} = \overrightarrow{a} \cdot \frac{\overrightarrow{b}}{|\overrightarrow{b}|}$. The vector projection means we multiply this by the unit vector $\hat{b}$.
  One can also use the dot product to find the angle between two vectors, if we know the lengths of these vectors. $\overrightarrow{a} \cdot \overrightarrow{b} = 0$ iff $\overrightarrow{a} \perp \overrightarrow{b}$.

- Cross product: This gives us a vector perpendicular to the two original vectors. $\overrightarrow{a} \times \overrightarrow{b}$ has length $|a|\,|b|\sin\theta$ and symbolically can be expressed as

$$\det \begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

  $|\overrightarrow{a}.(\overrightarrow{b} \times \overrightarrow{c})|$ gives the volume of the parallelepiped spanned by these three vectors. A cross product is zero if the two vectors are parallel. Also $|\overrightarrow{a} \times \overrightarrow{b}|$ equals the area of the parallelogram spanned by $\overrightarrow{a}$ and $\overrightarrow{b}$.

Lines, planes, vector functions: Sections 12.5–12.6, 13.1–13.2

- Symmetric and parametric equations for a line. Equation for a plane is
  $\overrightarrow{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$ where $\overrightarrow{n}$ is a normal and (x_0, y_0, z_0) is a point on plane.
  One way to get equation for a plane is to find two vectors lying on it (e.g. can do this if given three points on the plane), then take their cross product to get normal. Then choose constant so (x_0, y_0, z_0) satisfies the equation.

- Quadric surfaces: cylinder, ellipsoid, hyperboloid of one sheet, hyperboloid of two sheets, cone, elliptic paraboloid, the hyperbolic paraboloid. See the textbook pg. 830 for pictures of these.

- Vector functions: a vector $\overrightarrow{r}(t)$ which varies in time and its endpoint traces out a curve. Velocity is $\overrightarrow{r}'(t)$, obtained from differentiating coordinate-wise. Acceleration is $\overrightarrow{r}''(t)$.

- Dot and cross product satisfy product rules when differentiating (keep track of the order in cross product.)

- Can also integrate vector functions coordinate-wise.

Multivariable functions and calculus: Sections 14.1–14.7

- Level curves, contour plots, estimating partial derivatives given contour plots.

- Limits of functions: squeeze theorem, i.e. squeeze the function between two quantities tending to the same limit. Or try converting to polar coordinates.

- Partial derivatives: see how $z = f(x, y)$ or $f(x, y, z)$ changes when we vary only one variable and keep the others fixed.

- Implicit differentiation: when we want to find $\partial z/\partial x$ but are given $F(x, y, z) = 0$ instead of $z = f(x, y)$. So we differentiate $F(x, y, z) = 0$ with respect to x keeping y constant (but z depends on x).

- Tangent plane: for surface $z = f(x, y)$, tangent plane at point $(x_0, y_0, f(x_0, y_0))$ is
  $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$.

- In general, the tangent plane to level surface $F(x, y, z) = c$ (for constant c) at point (x_0, y_0, z_0), has normal $\langle F_x, F_y, F_z \rangle$. So tangent plane is $\langle F_x, F_y, F_z \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$ i.e.
  $F_x(x - x_0) + F_y(y - y_0) + F_z(z - z_0) = 0$. When $F(x, y, z) = f(x, y) - z$, this reduces to above case.

- Linear approximation: a small change in f can be approximated by small changes in each variable, times partial derivatives in each variable. For $f(x, y, z)$ it is $\Delta f \approx f_x \Delta x + f_y \Delta y + f_z \Delta z$, and for $f(x, y)$ there's no z term.

- Multivariable chain rule: differentiating $z = f(x, y)$ when x and y are also functions of variables.

- ∇f is a vector field with components given by the first order partial derivatives of f. It is perpendicular to level curves (for $f(x, y)$) or level surfaces (for $f(x, y, z)$).
  Directional derivative $D_{\overrightarrow{u}} f = \nabla f \cdot \frac{\overrightarrow{u}}{|\overrightarrow{u}|}$ is the rate of change of f in direction of $\overrightarrow{u}$.

- Critical points: we can find critical points by setting $\nabla f = 0$. Then use the second order derivative test to determine their local nature. If $D(x, y) := f_{xx} f_{yy} - f_{xy} f_{yx}$ then $D < 0$ gives a saddle point and $D > 0$ gives a max or min. Max when $f_{xx} < 0$ and min when $f_{xx} > 0$, analogous to one variable case.
  To determine the global behavior of f, i.e. absolute maxima and minima, we need to also check on the boundary of the domain.