Math 53 Discussion: Review for Midterm 1

Curves: Sections 10.1–10.4

- Parametric equations: $x = f(t)$, $y = g(t)$ for some functions f, g.
  Self-intersections of curves occur when two different values of t give same point (x, y).
  Derivative dy/dx at a point on the curve can be computed via $dy/dx = \frac{dy/dt}{dx/dt}$.

- Length of curves: length of small piece of curve is $\Delta s \approx \sqrt{\Delta x^2 + \Delta y^2}$ so length L of curve traced out from time t_1 to time t_2 is $L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$.

- Area under a curve: $A = \int_{x_1}^{x_2} y dx = \int_{t_1}^{t_2} y(t) \frac{dx}{dt} dt$. Want $x_1 < x_2$ so we're integrating from left to right, however it may be that t_1 is greater than t_2. We can't have lengths and areas be negative.

- Polar coordinates and curves: (r, θ) where r is the distance from the origin, θ is the angle from the positive x-axis. $x = r\cos\theta$, $y = r\sin\theta$, $x^2 + y^2 = r^2$ and $\tan\theta = y/x$.
  A polar curve is of the form $r = f(\theta)$.

- Polar areas: find the range of the angle θ in the area you are trying to compute, then $A = \int_{\theta_1}^{\theta_2} \frac{1}{2} r(\theta)^2 d\theta$. Sometimes θ_1, θ_2 were the intersection points of two polar curves, and we found the area between the two curves.

Vectors: Sections 12.1–12.4

- Vectors: how to manipulate vectors, such as adding and subtracting

- Dot product: This gives us a scalar. There are two ways of writing the dot product $\overrightarrow{a} \cdot \overrightarrow{b} = \sum_i a_i b_i$ and $|a|\ |b| \cos\theta$ where θ is the angle (at most π) between $\overrightarrow{a}$ and $\overrightarrow{b}$.
  Scalar and vector projections: projecting a vector $\overrightarrow{a}$ onto $\overrightarrow{b}$ means finding the component of $\overrightarrow{a}$ in the (unit) direction of $\overrightarrow{b}$. That is, $\text{comp}_{\overrightarrow{b}} \overrightarrow{a} = \overrightarrow{a} \cdot \frac{\overrightarrow{b}}{|\overrightarrow{b}|}$. The vector projection means we multiply this by the unit vector $\hat{b}$.
  Can also use dot product to find angles between vectors, if we know lengths of these vectors.
  Dot product is zero if two vectors are orthogonal.

- Cross product: This gives us a vector perpendicular to the two original vectors. $\overrightarrow{a} \times \overrightarrow{b}$ has length $|a|\ |b| \sin\theta$ and also is the determinant of 3×3 matrix. $|\overrightarrow{a}.(\overrightarrow{b} \times \overrightarrow{c})|$ gives the volume of parallelepiped spanned by these three vectors. Cross product is zero if two vectors are parallel. Length of cross product gives us the area of parallelogram spanned by $\overrightarrow{a}$ and $\overrightarrow{b}$.

Lines, planes, vector functions: Sections 12.5, 13.1–13.4

- Symmetric and parametric equations for a line. Equation for a plane is
  $\overrightarrow{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$ where $\overrightarrow{n}$ is a normal and (x_0, y_0, z_0) is a point on plane.
  One way to get equation for a plane is to find two vectors lying on it (e.g. can do this if given three points on the plane), then take their cross product to get normal. Then choose constant so (x_0, y_0, z_0) satisfies the equation.

- Vector functions: a vector $\overrightarrow{r}(t)$ which varies in time and traces out a curve. Velocity is $\overrightarrow{r}'(t)$, obtained from differentiating coordinate-wise. Acceleration is $\overrightarrow{r}''(t)$.
- Dot and cross product satisfy product rules when differentiating (keep track of the order in cross product.)
- Can also integrate vector functions coordinate-wise.

Multivariable functions and calculus: Sections 14.1–14.8

- Level curves, contour plots, estimating partial derivatives given contour plots.
- Limits of functions: squeeze theorem, i.e. squeeze the function between two quantities tending to the same limit.
- Partial derivatives: see how $z = f(x, y)$ or $f(x, y, z)$ changes when we vary only one variable and keep the others fixed.
- Implicit differentiation: when we want to find $\partial z/\partial x$ but are given $F(x, y, z) = 0$ instead of $z = f(x, y)$. So we differentiate $F(x, y, z) = 0$ with respect to x keeping y constant (but z depends on x).
- Tangent plane: for surface $z = f(x, y)$, tangent plane at point $(x_0, y_0, f(x_0, y_0))$ is $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$.
- In general, the tangent plane to level surface $F(x, y, z) = c$ (for constant c) at point (x_0, y_0, z_0), has normal $\langle F_x, F_y, F_z\rangle$. So tangent plane is $\langle F_x, F_y, F_z\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle =$
  $F_x(x - x_0) + F_y(y - y_0) + F_z(z - z_0) = 0$. When $F(x, y, z) = f(x, y) - z$, this reduces to above case.
- Linear approximation: change in f can be approximated by small changes in each variable, times partial derivatives in each variable. For $f(x, y, z)$ it is $\Delta f \approx f_x\Delta x + f_y\Delta y + f_z\Delta z$, and for $f(x, y)$ there's no z term.
- Multivariable chain rule: differentiating $z = f(x, y)$ when x and y are also functions of variables.
- ∇f is a vector with components given by the first order partial derivatives of f. It is perpendicular to level curves (2 dimensions) or level surfaces (three dimensions).
  Directional derivative $D_{\overrightarrow{u}} f = \nabla f \cdot \frac{\overrightarrow{u}}{|\overrightarrow{u}|}$ is rate of change of f in direction of $\overrightarrow{u}$.
- Critical points: we can find critical points by setting $\nabla f = 0$. Then use the second order derivative test to determine their local nature. To determine global behavior, i.e. absolute maxima and minima, we need to also check on the boundary of the domain.
- Lagrange multipliers: to find the extreme values of some function $f(x, y)$ subject to a condition $g(x, y) = c$, solve for $\nabla f = \lambda\nabla g$, $g(x, y) = c$. Then plug the values of x and y found to see what f is, and check what type of extremum it is.