Math 53 Discussion: Review for Midterm 2

Double integrals. §15.1 – 15.6

- Cartesian and polar coordinates. §15.1–15.4.
  - Analogous to an integral in 1D being the area under a curve, we can think of integrals in 2D as the volume under a surface. Integration can be done in the order $dx\ dy$ or $dy\ dx$. E.g. if the integral is in the order $dx\ dy$ over a region R, take the shadow of R on the y-axis to get the y-bounds, then see how x varies at each such y.
  - In polars, the area element is $dA = r\ dr\ d\theta$.
- Applications – mass, center of mass, moment of inertia, average value of a function. §15.5
  - Mass of lamina (infinitely thin sheet of paper) occupying 2D region R, with density ρ: integrate density over the region R to get mass equals $m = \int\int_R \rho\ dA$.
  - Center of mass $(\overline{x}, \overline{y})$ is obtained by finding the average x and y coordinates, weighted by the density.

$$\overline{x} = \frac{1}{m}\int\int_R x\rho\ dA, \quad \overline{y} = \frac{1}{m}\int\int_R y\rho\ dA$$

  - Moment of inertia: rotating lamina about x-axis (I_x), y-axis (I_y), and origin (I_0) involves integrating the (distance)^2 from the axis of rotation, over the lamina R of density ρ:

$$I_x = \int\int_R y^2\rho(x,y)\ dA, \quad I_y = \int\int_R x^2\rho(x,y)\ dA, \quad I_0 = \int_R (x^2+y^2)\rho(x,y)\ dA$$

  - Average value of a function f over region R is $\frac{1}{\text{area}(R)}\int\int_R f\ dA$.
- Surface area. §15.6. Surface area of a surface obtained as a graph $z = f(x,y)$ over region R in the xy-plane:

$$\int\int_R \sqrt{{f_x}^2 + {f_y}^2 + 1}\ dx\ dy$$

Triple integrals. §15.7–15.9

- Triple integrals involve integrating a function f over a 3D solid. We can think of this as finding the mass of such a solid with density f. (But a general triple integral can be negative.)
- Method: e.g. suppose the area element is $dx\ dy\ dz$, and we want to integrate over a solid E. First find the shadow of E on the yz-plane (this will be the slice parallel to the yz-plane where E is "widest" and not necessarily the same as setting $x = 0$ in the equation for E). Express the region for this shadow as you would a double integral to get the y and z bounds. Then at each point (y, z) in this region, see how x varies as a function of y and z.
- Cylindrical coordinates (r, θ, z): these are 2D polar coordinates along with the third coordinate being z. The volume element is $dV = r\ dr\ d\theta\ dz$.
- Spherical coordinates (ρ, θ, ϕ): here ρ is the distance of the point to the origin, θ is the same as in cylindrical, i.e. the angle from the positive x-axis when projected to the xy-plane, and ϕ is the angle from the positive z-axis. Note that ϕ takes values between 0 and π.
- Applications: analogous to the formulas above. Now density is a function of three variables, and for average value of a function we divide by volume instead of area.

Change of variables using the Jacobian. §15.10

- Allows us to evaluate integrals by changing variables so the domain is nicer. Examples include polar coordinates in 2D, and spherical and cylindrical coordinates in 3D.
- Method in 2D (3D is similar):
  1. Change from region R in xy-plane to region S in uv-plane by $x = x(u,v), y = y(u,v)$. E.g. see where the boundaries of R map to by finding equations describing them in x and y, then plugging in $x(u,v)$ and $y(u,v)$ to find the corresponding equations in u and v of the boundaries for S.
  2. The area element dA becomes $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\ dv$. (One way to remember what goes on top and on bottom is to think of $\partial(u,v)$ as "canceling" $du\ dv$ to give $dx\ dy$. Mathematically this isn't exactly what's happening, but it's a way to remember the order.)
  3. Here $\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$ is called the Jacobian of the transformation. So the first order partials of x go in the first row and of y go in the second row. (We have a 3×3 matrix in 3D.) We take the absolute value of this determinant when evaluating the integral.
  4. Evaluate the double integral now in terms of u and v.
- If we're given u and v in terms of x and y there are two ways to find $\frac{\partial(x,y)}{\partial(u,v)}$:
  1. Solve for x and y as functions of u and v and find the partial derivatives directly.
  2. Find $\frac{\partial(u,v)}{\partial(x,y)}$ and then take the reciprocal to get what we want, i.e. $\frac{\partial(x,y)}{\partial(u,v)}$. In other words,

$$\frac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}} = \frac{\partial(x,y)}{\partial(u,v)}$$

- To summarize:
  1. Find the new region S in the uv-plane and the new bounds describing S.
  2. The area element becomes $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\ dv$.
  3. Write the function we're integrating, $f(x,y)$, in terms of u and v as $f(x(u,v), y(u,v))$.
  4. Evaluate the integral now written in terms of u and v, using the usual tools for multiple integrals.

Vector fields, line integrals. §16.1–16.2

- A vector field is a function on the plane or in three-dimensional space. It's a function which assigns a vector at each point in its domain. For example, $\overrightarrow{F}(x,y,z)$ could describe the direction the wind is traveling at (x,y,z).
- Line integrals allow us to integrate a scalar function or vector function over a curve C, by parametrizing the curve and reducing to an integral in one variable.
- Integrating a scalar function (e.g. if C describes a thin wire with linear density $f(x,y,z)$ then we integrate f over C to get the mass of the wire):

$$\int_C f\ ds$$

  where s is arc length.
- Integrating a vector field $\overrightarrow{F}$ to get work done (dot product with $d\overrightarrow{r} = \langle dx, dy\rangle$) or flux through the curve (dot product with $\overrightarrow{n}ds = \langle dy, -dx\rangle$ where $\overrightarrow{n}$ is the unit normal and ds is infinitesimal arc length):

$$\int_C \overrightarrow{F} \cdot d\overrightarrow{r}, \quad \text{or for flux: } \int_C \overrightarrow{F} \cdot \overrightarrow{n} ds$$

- Method:
  - Parametrize the curve C as $\overrightarrow{r}(t) = \langle x(t), y(t), z(t)\rangle$, or only in x and y if it's a planar curve.
  - Determine what type of integral you're doing: if you're integrating a scalar then take the **magnitude** $|r'(t)|$ to get
$$ds = |r'(t)|\, dt$$
    If you're integrating a vector field, then take the **dot product** of $\overrightarrow{F}$ with
$$d\overrightarrow{r} = \overrightarrow{r}'(t)dt, \text{ or } \overrightarrow{n}ds = \big\langle y'(t), -x'(t)\big\rangle\, dt$$
  - Plug everything into the integral to get an integral in t, which can then be evaluated.

The fundamental theorem of line integrals. §16.3

- For conservative vector fields $\overrightarrow{F} = \nabla f$ only we have that line integrals are independent of path:
$$\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = \int_C \nabla f \cdot d\overrightarrow{r} = f(\overrightarrow{r}(b)) - f(\overrightarrow{r}(a))$$
- In particular, if we have a closed curve, meaning $\overrightarrow{r}(a) = \overrightarrow{r}(b)$, the line integral is zero.
- If $\overrightarrow{F} = \langle P, Q\rangle$ and $\partial Q/\partial x = \partial P/\partial y$ on a simply connected region, then $\overrightarrow{F}$ is conservative on that region. This is a special case of checking that $\text{curl}(\overrightarrow{F}) = \overrightarrow{0}$ on a simply connected region.

Green's theorem. §16.4

- C is a simple closed, positively oriented curve, enclosing region D, and $\overrightarrow{F} = \langle P, Q\rangle$ has continuous partial derivatives on an open region containing D then:
$$\int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\, dA = \int_C \overrightarrow{F} \cdot d\overrightarrow{r} = \int_C P dx + Q dy$$

Curl and divergence. §16.5

- Let $\overrightarrow{F} = \langle P, Q, R\rangle$. Then
$$\text{curl}(\overrightarrow{F}) = \nabla \times \overrightarrow{F} = \det\begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{pmatrix}$$
- Thinking again of $\overrightarrow{F}$ as fluid flow, $\text{curl}(\overrightarrow{F})$ points in the direction of the axis of rotation of the fluid. A larger magnitude for curl means faster rotation. $\text{curl}(\overrightarrow{F}) = \overrightarrow{0}$ means $\overrightarrow{F}$ is irrotational.
- $$\text{div}(\overrightarrow{F}) = \nabla \cdot \overrightarrow{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$
- Thinking of $\overrightarrow{F}$ as describing fluid flow, divergence of $\overrightarrow{F}$ at a point measures the amount of fluid flowing in or out at that point. Positive divergence means the net flow is outwards (source) and negative divergence means the net flow is inwards (sink). $\text{div}(\overrightarrow{F}) = 0$ means $\overrightarrow{F}$ is incompressible.

Parametrized surfaces. §16.6

- Method:

1. Parametrize S as $\overrightarrow{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$, find the bounds for u, v. Here are some examples.

   Graph of a function $z = f(x,y)$: x and y are the parameters.

   $$\overrightarrow{r}(x,y) = \langle x, y, f(x,y)\rangle$$

   The domain will be the "shadow" of the surface in the xy-plane.

   A sphere of radius a: ϕ and θ are the parameters.

   $$\overrightarrow{r}(\phi,\theta) = \langle a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\phi\rangle$$

   You determine the bounds on ϕ and θ depending on what portion of the surface of the sphere you want.

   A cylinder of radius a along z-axis: the height z and the polar angle θ are the parameters.

   $$\overrightarrow{r}(\theta,z) = \langle a\cos\theta, a\sin\theta, z\rangle$$

2. Compute $\overrightarrow{r}_u \times \overrightarrow{r}_v$. Then $dS = |\overrightarrow{r}_u \times \overrightarrow{r}_v|\; du\; dv$. Whatever your parameters u and v are, you only multiply the magnitude of the cross product by $du\; dv$. If your parameters are r and θ, you will only multiply by $dr\; d\theta$. However, if your parameters are x and y and you get a double integral involving $dx\; dy$ and then want to convert into polars, you will need to use $r\; dr\; d\theta$.

   Graph of a function:

   $$\overrightarrow{r}_x \times \overrightarrow{r}_y = \langle -f_x, -f_y, 1\rangle$$
   $$dS = \sqrt{f_x^2 + f_y^2 + 1}$$

   A sphere of radius a:

   $$\overrightarrow{r}_\phi \times \overrightarrow{r}_\theta = \left\langle a^2\sin^2\phi\cos\theta, a^2\sin^2\phi\sin\theta, a^2\sin\phi\cos\phi\right\rangle$$
   $$dS = a^2\sin\phi\; d\phi\; d\theta$$

   A cylinder of radius a along z-axis:

   $$\overrightarrow{r}_\theta \times \overrightarrow{r}_z = \langle a\cos\theta, a\sin\theta, 0\rangle$$
   $$dS = a\; dz\; d\theta$$

- Surface area from §15.6 is a special case of this. In that section, we were dealing only with surfaces that were graphs of functions (not all surfaces are graphs of functions, e.g. a sphere). The surface area is

  $$\int\!\!\int_S 1\; dS = \int\!\!\int_D \sqrt{f_x^2 + f_y^2 + 1}\; dx\; dy$$

  where S is the surface and D is the domain in x and y.

- The Jacobian from §15.10 is also a special case of this. There our surface S was in the flat plane, so the parametrization was of the form $\overrightarrow{r}(u,v) = \langle x(u,v), y(u,v), 0\rangle$. The area element $dS = |\overrightarrow{r}_u \times \overrightarrow{r}_v|\; du\; dv$ is just the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$.

  $$\det\begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ x_u & y_u & 0 \\ x_v & y_v & 0 \end{pmatrix} = \langle 0, 0, x_u y_v - x_v y_u\rangle$$

  which has magnitude $|x_u y_v - x_v y_u| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|$.