Math 53 Discussion: Review for Final

Parametrized surfaces, Surface integrals. §16.6–16.7

- We have seen two types of surface integrals. We can integrate a scalar function or we can integrate a vector function.
- Method:
  1. Parametrize S as $\overrightarrow{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$, find the bounds for u, v. Here are some examples.

     Graph of a function $z = f(x,y)$: x and y are the parameters.

     $$\overrightarrow{r}(x,y) = \langle x, y, f(x,y)\rangle$$

     The domain will be the "shadow" of the surface in the xy-plane.

     A sphere of radius a: ϕ and θ are the parameters.

     $$\overrightarrow{r}(\phi,\theta) = \langle a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\phi\rangle$$

     You determine the bounds on ϕ and θ depending on what portion of the surface of the sphere you want.

     A cylinder of radius a along z-axis: the height z and the polar angle θ are the parameters.

     $$\overrightarrow{r}(\theta,z) = \langle a\cos\theta, a\sin\theta, z\rangle$$

  2. Compute $\overrightarrow{r}_u \times \overrightarrow{r}_v$. Then $\overrightarrow{n}\,dS = \pm\overrightarrow{r}_u \times \overrightarrow{r}_v\; du\, dv$. Whatever your parameters u and v are, you only multiply the cross product by $du\, dv$. If your parameters are r and θ, you will only multiply by $dr\, d\theta$. However, if your parameters are x and y and you get a double integral involving $dx\, dy$ and then want to convert into polars, you will need to use $r\, dr\, d\theta$.

     Graph of a function:

     $$\overrightarrow{r}_x \times \overrightarrow{r}_y = \langle -f_x, -f_y, 1\rangle$$

     A sphere of radius a:

     $$\overrightarrow{r}_\phi \times \overrightarrow{r}_\theta = \left\langle a^2\sin^2\phi\cos\theta, a^2\sin^2\phi\sin\theta, a^2\sin\phi\cos\phi\right\rangle$$

     Or a geometric argument: the normal is $\frac{1}{a}\langle x,y,z\rangle$ and the area element on the sphere is $dS = a^2\sin\phi\; d\phi\; d\theta$ so multiplying these together gives

     $$d\overrightarrow{S} = a\sin\phi\,\langle x,y,z\rangle\; d\phi\; d\theta$$

     where x, y and z are in terms of ϕ and θ.

     A cylinder of radius a along z-axis:

     $$\overrightarrow{r}_\theta \times \overrightarrow{r}_z = \langle a\cos\theta, a\sin\theta, 0\rangle$$

     Or a geometric argument: the normal is $\frac{1}{a}\langle x,y,0\rangle$ and $dS = a\; dz\; d\theta$ so multiplying together gives

     $$d\overrightarrow{S} = \langle x,y,0\rangle\, dz\; d\theta = \langle a\cos\theta, a\sin\theta, 0\rangle\, dz\; d\theta$$

3. For integrals of a **scalar function** f: The integrand will be $f\ dS$ so we take the **magnitude**

$$dS = |\overrightarrow{r}_u \times \overrightarrow{r}_v|\ du\ dv$$

For integrals of a **vector function** $\overrightarrow{F}$: The integrand will be $\overrightarrow{F} \cdot d\overrightarrow{S}$ so we take the **dot product with the vector**

$$d\overrightarrow{S} = \pm \overrightarrow{r}_u \times \overrightarrow{r}_v\ du\ dv$$

- Surface area from §15.6 is a special case of this. In that section, we were dealing only with surfaces that were graphs of functions (not all surfaces are graphs of functions, e.g. a sphere). We were also dealing only with integrals of scalar functions, namely integrating 1. So the dS in that case is $\sqrt{f_x^2 + f_y^2 + 1}$ and the surface area is

$$\int\int_S 1\ dS = \int\int_D \sqrt{f_x^2 + f_y^2 + 1}\ dx\ dy$$

where S is the surface and D is the domain in x and y.

- The Jacobian from §15.10 is also a special case of this. There our surface S was in the flat plane, so the parametrization was of the form $\overrightarrow{r}(u, v) = \langle x(u, v), y(u, v), 0\rangle$. In that section we looked at integrals of scalar functions so the area element is $dS = |\overrightarrow{r}_u \times \overrightarrow{r}_v|\ du\ dv$. This is just the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$.

$$\det \begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ x_u & y_u & 0 \\ x_v & y_v & 0 \end{pmatrix} = \langle 0, 0, x_u y_v - x_v y_u\rangle$$

which has magnitude $|x_u y_v - x_v y_u| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|$.

Stokes' theorem. §16.8

- Taking compatible orientations on **closed curve** C bounding surface S (meaning if we walk around C with our head in the direction of the normal, the surface is to the left):

$$\int\int_S \text{curl}\,\overrightarrow{F} \cdot d\overrightarrow{S} = \int_C \overrightarrow{F} \cdot d\overrightarrow{r}$$

Divergence theorem. §16.9

- E is a solid enclosed by a **closed surface** S, with outward normal taken on S:

$$\int\int\int_E \text{div}(\overrightarrow{F})\ dV = \int\int_S \overrightarrow{F} \cdot d\overrightarrow{S}$$