Math 53 Discussion

Practice Problems: 16.1–16.2, vector fields, line integrals

1) [Question 2, Worksheet 24] Sketch the vector fields

$$\begin{aligned}
\overrightarrow{F}(x, y) &= \frac{x}{\sqrt{x^2 + y^2}}\,\hat{\mathbf{i}} + \frac{y}{\sqrt{x^2 + y^2}}\,\hat{\mathbf{j}} \\
\overrightarrow{G}(x, y) &= \frac{-y}{\sqrt{x^2 + y^2}}\,\hat{\mathbf{i}} + \frac{x}{\sqrt{x^2 + y^2}}\,\hat{\mathbf{j}}
\end{aligned}$$

by first showing they have unit length and are perpendicular everywhere they are defined. (In fact, the *flow lines* of the vector field $\overrightarrow{G}$ are circles centered at the origin, parametrized with unit speed. See exercise 35 in §16.1 on page 1062 for a definition of flow lines.)

2) Sketch the contour plot and gradient vector field of $f(x, y) = xy$. For example, you could start by sketching the contours $f(x, y) = 1$ and $f(x, y) = 2$.

3) Find $\int_C z\, ds$ where C is the helix $(\cos t, \sin t, t)$ for $0 \le t \le \pi$.

4) Find $\int_C xy\, dx + (x-y)\, dy$, where $C =$ line segments from $(0,0)$ to $(2,0)$ and $(2,0)$ to $(3,2)$.

5) Evaluate $\int_C \overrightarrow{F} \cdot d\overrightarrow{r}$ where $\overrightarrow{F} = x^2y^3\ \hat{\mathbf{i}} - y\sqrt{x}\ \hat{\mathbf{j}}$ and $\overrightarrow{r}(t) = t^2\ \hat{\mathbf{i}} - t^3\ \hat{\mathbf{j}}$ for $0 \le t \le 1$.

Answers. 1) $\overrightarrow{F}$ is the unit vector field that points radially outward, and $\overrightarrow{G}$ is the vector field that is tangent to circles of radius a parametrized by $x = a\cos(t/a), y = a\sin(t/a)$. 2) Vector field arrows perpendicular to contour lines, which are hyperbolas. Arrows point in direction of increasing f. 3) $\frac{\pi^2\sqrt{2}}{2}$. 4) $\frac{17}{3}$. 5) $-\frac{59}{105}$.