Math 53 Discussion

Practice Problems (from textbook and Math 53 worksheets):
Area and arc length of parametric curves

1) When a parametric curve comes from the graph of a function \( y = f(x) \) for \( a \leq x \leq b \), show that the formula for arc-length gives
\[
\int_a^b \sqrt{1 + f'(x)^2} \, dx
\]

2) [Worksheet 2, Problem 1] Let \( C \) be the curve \( x = 2 \cos t, y = \sin t \).
   • What kind of curve is this?
   • Find the slope of the tangent line to the curve when \( t = 0, t = \pi/4, \) and \( t = \pi/2 \).
   • Find the area of the region enclosed by \( C \). (Hint: \( \sin^2 t = (1 - \cos 2t)/2 \).)
3) Find the area of the region enclosed by the curve \( x(t) = 1 - t, y(t) = e^t \) and the vertical lines \( x = 0, x = 2 \).

4) Find the arc length of the curve \( x(t) = e^t + e^{-t}, y(t) = 2t - 5 \) for \( 0 \leq t \leq 3 \).

**Answers:** 1) Use the parametrization \((t, f(t))\). 2) Ellipse. Slopes are infinite, \(-1/2, 0\). Area is \(2\pi\); it helps to compute a quarter of the area and then multiply by 4. Note that the bounds on \( x \) always go from smaller \( x \) to larger \( x \), however when we change variables to \( t \) using the parametrization, it may happen that the lower bound on \( t \) is larger than the upper bound on \( t \). That’s okay - we get a positive answer which is what we expect for area. 3) \( \int_{-1}^{1} e^t (-dt) = e - \frac{1}{e} \). 4) \( \int_{0}^{3} \sqrt{(e^t - e^{-t})^2 + 4} \, dt = e^3 - e^{-3} \).