First-Order Modal Logic

Notes by R.J. Buehler
Based on *First-Order Modal Logic* by Fitting and Mendelsohn

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Preface

What follows are my personal notes created during an independent study under Professor Holliday as a graduate student. This text is made available as a convenient reference, set of notes, and summary, but without even the slightest hint of a guarantee that everything contained within is factual and correct. This said, if you find an error, I would much appreciate it if you let me know so that it can be corrected.

Chapter 1

Quantified Modal Logic

1.1 First-Order Formulas

In addition to the standard connectives and quantifiers, we assume that, for every n, an infinite stock of n-place relation symbols $(P_1^n, P_2^n, \dots)$ is available and that an infinite stock of variables $(v_1, v_2, \dots)$ is also available .

Definition: Atomic Formula
An expression of the form $R(x_1, x_2, \dots, x_n)$ for some relation R of arity n and sequence of variables $x_1, x_2, \dots, x_n$.

Definition: First-Order Modal Formulas
The set of first-order formulas and free variable occurrences are as follows:

i. Every atomic formula is a first-order formula; every occurrence of a variable in an atomic formula is a free variable.

ii. If ϕ is a first-order formula, so is $\neg\phi$; every free variable of ϕ is a free variable of $\neg\phi$.

iii. If ϕ, ψ are first-order formula and $\circ$ is a binary connective, $(\phi \circ \psi)$ is a first-order formula; the free variables of $(\phi \circ \psi)$ are all those of ϕ and ψ.

iv. If ϕ is a first-order formula, then $\Box\phi$ and $\Diamond\phi$ are also first-order formula; the free variables of $\Box\phi$ and $\Diamond\phi$ are all those of ϕ.

v. If ϕ is a first-order formula and v a variable, then $\forall v(\phi)$ and $\exists v(\phi)$ are first-order formulas; the free variables of $\forall v(\phi)$ and $\exists v(\phi)$ are all those of ϕ except v.

Any variables occuring in a formula which are not free are said to be *bound*.

Definition: Sentence
A first-order formula without free variables; also known as a *closed formula*.

1.2 Necessity De Re and De Dicto

Example 1. *The sentence 'Something necessarily exists' can be parsed in two different ways:*

It is necessary that there is something which exists.
There is something which has the property of necessary existence.

Assuming a non-empty domain, the first is trivially true; the later, however, is highly non-trivial.

In general, the reading where the property (in this case, necessity) applies to a proposition (*dictum*) is said to be the *de dicto* interpretation (the first reading above). Similarly, the reading where the property applies instead to an object (*re*) is said to be the *de re* interpretation (the second reading above). As it turns out, the *de re* - *de dicto* distinction is readily representable in our current system as a scope distinction:

De Dicto	$\Box\forall x F(x)$
De Re	$\forall x \Box F(x)$

1.3 Is Quantified Modal Logic Possible?

W.V.O. Quine has argued, quite famously, that quantified modal logic is an incoherent project; his reasons center on the three grades of modal involvement or three interpretations for the $\Box$ symbol:

i. The least troublesome interpretation of $\Box$ is to take it as a meta-linguistic predicate attaching to sentences

   (a) Quine goes on to say that to be necessarily true is simply to be a theorem of something sufficiently close to logic, although it's not clear than this interpretation is necessary.

   (b) This grade doesn't allow for the iteration of $\Box$

ii. The second grade–and the one used in propositional modal logic–is to take $\Box$ as an operator attaching to sentences; it blurs the use-mention distinction.

   (a) Note, however, that (perhaps unsurprisingly) how an object is specified affects truth values:

Example 2.

$$\Box(\text{ The number of planets} > 7)$$

$$\Box(8 > 7)$$

The first is false while the latter is true–despite 'The number of planets' being 8.

iii. At grade 3, $\Box$ attaches to open formulas.

   (a) The problem, as Quine argues, is that–as we saw in grade 2–*how* you specify the object matters!

Example 3.

$$\Box(x > 7)$$

Should the above be interpreted along the lines of the top or bottom sentences in the example above?

(b) In grade 2, $\Box$ depends on the ways things are picked out, but grade 3 requires that $\Box$ be independent of the way things are picked out! It'll never work!

This seems true, but note that the only kind of necessity invoked throughout is *de dicto*! The problem disappears once the move to *de re* is made.

1.4 Quantifying

An interesting problem arises with the domain of objects in first-order modal logic; namely, how does the domain relate to the various possible worlds under consideration? It's natural to consider possible worlds where some object a from the actual world occurs in the possible world as well–but must every object from the actual world occur in that possible world? Could there be objects in that possible world which aren't in the actual world? Noting that the quantifiers $\forall$ and $\exists$ are assumed to quantify only over the domain of the current possible world, it's easy to see that how these questions are answered greatly impacts the resulting logic; consider:

$$\forall x \phi(x) \to \phi(y)$$

is valid if every possible world has exactly the same set of objects that it can refer to (if every object of this world is ϕ and no possible worlds have any different objects, then it doesn't matter what y is). This scheme is called *constant domain.* If, however, the objects in each world's domain are allowed to vary–*variable domain*–then the formula above is not valid; an object which doesn't occur in the current world's domain may be $\neg\phi$. In either case, the *domain of the model* is the union of the domain of every possible world in the model.

Why are free variables allowed to select outside the current world, but quantifiers are stuck in the world?

These two schemes translate to two different kinds of quantification; in a constant domain model, the domain usually allows non-actual objects (otherwise, every object necessarily exists). This move preserves the classical quantifier rules, but at the cost of referencing non-existent entities. Quantification in this scheme is called *possibilist quantification.*

The alternative is to adopt a variable domain model and quantify over only the objects in the possible world under consideration; unfortunately, this invalidates many classical inferences. This scheme is, rather appropriately, called *actualist quantification.*

1.4.1 Constant Domain Models

Definition: Augmented Frame
A structure $\langle g, \mathcal{R}, \mathcal{D} \rangle$ is a *constant domain augmented frame* if and only if $\langle g, \mathcal{R} \rangle$ is a frame (g a non-empty set of worlds, $\mathcal{R}$ a binary accessibility relation on g) and $\mathcal{D}$ is a non-empty set called the *domain* of the frame.

Under the constant domain scheme, all quantifiers range over the set $\mathcal{D}$–no matter the world under consideration.

Definition: Interpretation
A function ℓ is an interpretation of a constant domain augmented frame $\langle g, \mathcal{R}, \mathcal{D} \rangle$ if ℓ assigns to each n-place relation symbols R and each possible world $\gamma \in g$, some n-place relation on the domain $\mathcal{D}$ of the frame. If some n-tuple of elements of $\mathcal{D}$, $\langle d_1, d_2, \dots, d_n \rangle$, is a member of $\ell(R, \gamma)$, we write $\langle d_1, d_2, \dots, d_n \rangle \in \ell(R, \gamma)$.

Definition: Model
A constant domain first-order model is a structure $\mathcal{M} = \langle g, \mathcal{R}, \mathcal{D}, \ell \rangle$ where $\langle g, \mathcal{R}, \mathcal{D} \rangle$ is a constant domain augmented frame and ℓ is an interpretation of it. $\mathcal{M}$ is said to be a constant domain first-order model for a modal logic $\mathcal{L}$ if $\langle g, \mathcal{R} \rangle$ is an $\mathcal{L}$-frame in the propositional sense.

Truth in a Constant Domain Model

Definition: Valuation
Let $\mathcal{M} = \langle g, \mathcal{R}, \mathcal{D}, \ell \rangle$ be a constant domain first-order model. A valuation in the model $\mathcal{M}$ is a mapping v that assigns to each free variable x some member $v(x)$ of the domain $\mathcal{D}$ of the model.

Definition: Variant
Let v and w be two valuations. w is said to be an x-variant of v if w and v are the same except possibly at x.

Definition: Truth in a Model
Let $\mathcal{M} = \langle g, \mathcal{R}, \mathcal{D}, \ell \rangle$ be a constant domain first-order modal model. For each $\gamma \in g$ and each valuation v of $\mathcal{M}$:

i. If R is an n-place relation symbol, $\mathcal{M}, \gamma \Vdash_v R(x_1, x_2, \ldots, x_n)$ provided $\langle v(x_1), v(x_2), \ldots, v(x_n) \rangle \in \ell(R, \gamma)$

ii. $\mathcal{M}, \gamma \Vdash_v \neg\phi$ if and only if $\mathcal{M}, \gamma \nVdash_v \phi$

iii. $\mathcal{M}, \gamma \Vdash_v \phi \wedge \psi$ if and only if $\mathcal{M}, \gamma \Vdash_v \phi$ and $\mathcal{M}, \gamma \Vdash_v \psi$

iv. $\mathcal{M}, \gamma \Vdash_v \Box\phi$ if and only if for every $\delta \in g$, if $\gamma\mathcal{R}\delta$, then $\mathcal{M}, \delta \Vdash_v \phi$

v. $\mathcal{M}, \gamma \Vdash_v \Diamond\phi$ if and only if for some $\delta \in g$ such that $\gamma\mathcal{R}\delta$, $\mathcal{M}, \delta \Vdash_v \phi$

vi. $\mathcal{M}, \gamma \Vdash_v \forall x\phi$ if and only if for every x-variant w of v, $\mathcal{M}, \gamma \Vdash_w \phi$

vii. $\mathcal{M}, \gamma \Vdash_v \mathcal{E}x\phi$ if and only if for some x-variant w of v, $\mathcal{M}, \gamma \Vdash_w \phi$

► **Theorem 1.4.1.** *Suppose that $\mathcal{M} = \langle \mathrm{g}, \mathcal{R}, \mathcal{D}, \ell \rangle$ is a constant domain first-order modal model, $\gamma \in \mathrm{g}$, v and w two valuations on $\mathcal{M}$, and ϕ a formula. If v and w agree on all the free variables of ϕ, then*

$$\mathcal{M}, \gamma \Vdash_v \phi \text{ if and only if } \mathcal{M}, \gamma \Vdash_w \phi$$

► **Theorem 1.4.2.** *Suppose that $\mathcal{M} = \langle \mathrm{g}, \mathcal{R}, \mathcal{D}, \ell \rangle$ is a constant domain first-order modal model, $\gamma \in \mathrm{g}$, and v, w are two valuations on $\mathcal{M}$. Suppose further that $\phi(x)$ is a formula that may contain free occurrences of x, y doesn't appear in $\phi(x)$ at all, and $\phi(y)$ is the result of replacing all occurrences of x by y. Finally, suppose that v, w agree on all the free-occurrences of ϕ except x where $v(x) = w(y)$. Then,*

$$\mathcal{M}, \gamma \Vdash_v \phi(x) \text{ if and only if } \mathcal{M}, \gamma \Vdash_w \phi(y)$$

Definition: True at a World
Let $\mathcal{M} = \langle g, \mathcal{R}, \mathcal{D}, \ell \rangle$ be a constant domain first-order modal model and $\gamma \in g$. For a *sentence* ϕ, if $\mathcal{M}, \gamma \Vdash_v \phi$ for some valuation v on $\mathcal{M}$, then $\mathcal{M}, \gamma \Vdash_v \phi$ for every valuation v on $\mathcal{M}$. In this case, ϕ is said to be *true at* γ, symbolized $\mathcal{M}, \gamma \Vdash \phi$.

Definition: Model Valid
A sentence ϕ is said to be *valid in the model* if and only if it is true at every world in the model.

Definition: Frame Valid
A sentence ϕ is said to be *valid in the frame* if and only if it is valid in every model based on the frame.

1.4.2 Varying Domain Models

Definition: Augmented Frame
A structure $\langle g, \mathcal{R}, \mathcal{D} \rangle$ is a *varying domain augmented frame* if and only if $\langle g, \mathcal{R} \rangle$ is a frame and $\mathcal{D}$ is a function mapping members of g to non-empty sets. $\mathcal{D}$ is called the *domain function* and, for some $\gamma \in g$, $\mathcal{D}(\gamma)$ is called the *domain of* γ.

Under the varying domain scheme, all quantifiers range over the domain of the world under consideration; a constant domain model, then, can be simulated as simply a varying domain model which happens not to have any of the domains change.

Already, however, a problem occurs; in a formula like $\Box[p(x) \vee \neg p(x)]$, the standard valuation scheme won't suffice since the objects in different worlds differ. Once solution is to make the formula undefined or 'neither true, nor false' when an object doesn't exist in the domain of a world. The scheme adopted here is to instead treat talk of the object as meaningful, and thus have truth values for p applied to it.

This kind of seems like a constant domain model...

Definition: Frame Domain
Let $\mathcal{F} = \langle g, \mathcal{R}, \mathcal{D} \rangle$ be a varying domain augmented frame. The *domain of the frame* is the set $\cup\{\mathcal{D}(\gamma) : \gamma \in g\}$, abbreviated $\mathcal{D}(\mathcal{F})$.

Definition: Interpretation
A function ℓ is an interpretation of a varying domain augmented frame $\langle g, \mathcal{R}, \mathcal{D} \rangle$ if ℓ assigns to each n-place relation symbols R and each possible world $\gamma \in g$, some n-place relation on the domain of the frame, $\mathcal{D}(\mathcal{F})$. If some n-tuple of elements of $\mathcal{D}$, $\langle d_1, d_2, \ldots, d_n \rangle$, is a member of $\ell(R, \gamma)$, we write $\langle d_1, d_2, \ldots, d_n \rangle \in \ell(R, \gamma)$.

Definition: Model
A varying domain first-order model is a structure $\mathcal{M} = \langle g, \mathcal{R}, \mathcal{D}, \ell \rangle$ where $\langle g, \mathcal{R}, \mathcal{D} \rangle$ is a varying domain augmented frame and ℓ is an interpretation of it. $\mathcal{M}$ is said to be a varying domain first-order model for a modal logic $\mathcal{L}$ if $\langle g, \mathcal{R} \rangle$ is an $\mathcal{L}$-frame in the propositional sense.

Truth in a Varying Domain Model

Definition: Valuation
Let $\mathcal{M} = \langle g, \mathcal{R}, \mathcal{D}, \ell \rangle$ be a varying domain first-order model. A valuation in the model $\mathcal{M}$ is a mapping v that assigns to each free variable x some member $v(x)$ of the domain of the model, $\mathcal{D}(\mathcal{M})$.

Definition: Variant
Let v and w be two valuations. w is said to be an x-variant of v if w and v are the same except possibly at x; w is an x-variant of v at a world γ if and only if w is an x-variant of v and $w(x)$ is a member of $\mathcal{D}(\gamma)$.

Definition: Truth in a Model
Let $\mathcal{M} = \langle g, \mathcal{R}, \mathcal{D}, \ell \rangle$ be a varying domain first-order modal model. For each $\gamma \in g$ and each valuation v of $\mathcal{M}$:

i. If R is an n-place relation symbol, $\mathcal{M}, \gamma \Vdash_v R(x_1, x_2, \ldots, x_n)$ provided $\langle v(x_1), v(x_2), \ldots, v(x_n) \rangle \in \ell(R, \gamma)$

ii. $\mathcal{M}, \gamma \Vdash_v \neg\phi$ if and only if $\mathcal{M}, \gamma \nVdash_v \phi$

iii. $\mathcal{M}, \gamma \Vdash_v \phi \wedge \psi$ if and only if $\mathcal{M}, \gamma \Vdash_v \phi$ and $\mathcal{M}, \gamma \Vdash_v \psi$

iv. $\mathcal{M}, \gamma \Vdash_v \Box\phi$ if and only if for every $\delta \in g$, if $\gamma\mathcal{R}\delta$, then $\mathcal{M}, \delta \Vdash_v \phi$

v. $\mathcal{M}, \gamma \Vdash_v \Diamond\phi$ if and only if for some $\delta \in g$ such that $\gamma\mathcal{R}\delta$, $\mathcal{M}, \delta \Vdash_v \phi$

vi. $\mathcal{M}, \gamma \Vdash_v \forall x\phi$ if and only if for every x-variant w of v at γ, $\mathcal{M}, \gamma \Vdash_w \phi$

vii. $\mathcal{M}, \gamma \Vdash_v \mathcal{E}x\phi$ if and only if for some x-variant w of v at γ, $\mathcal{M}, \gamma \Vdash_w \phi$

► **Theorem 1.4.3.** *Suppose that $\mathcal{M} = \langle \mathrm{g}, \mathcal{R}, \mathcal{D}, \ell \rangle$ is a varying domain first-order modal model, $\gamma \in \mathrm{g}$, v and w two valuations on $\mathcal{M}$, and ϕ a formula. If v and w agree on all the free variables of ϕ, then*

$$\mathcal{M}, \gamma \Vdash_v \phi \text{ if and only if } \mathcal{M}, \gamma \Vdash_w \phi$$

► **Theorem 1.4.4.** *Suppose that $\mathcal{M} = \langle \mathrm{g}, \mathcal{R}, \mathcal{D}, \ell \rangle$ is a varying domain first-order modal model, $\gamma \in \mathrm{g}$, and v, w are two valuations on $\mathcal{M}$. Suppose further that $\phi(x)$ is a formula that may contain free occurrences of x, y doesn't appear in $\phi(x)$ at all, and $\phi(y)$ is the result of replacing all occurrences of x by y. Finally, suppose that v, w agree on all the free-occurrences of ϕ except x where $v(x) = w(y)$. Then,*

$$\mathcal{M}, \gamma \Vdash_v \phi(x) \text{ if and only if } \mathcal{M}, \gamma \Vdash_w \phi(y)$$

1.5 Existence Relativization

Which then of these two model theories should be taken as primary? It turns out, rather interestingly, that both amount to the same thing in a very formal sense. It's easy to see how varying domain models can be used to represent constant domain ones, but the reverse is also true if an existence predicate is available for each world.

Definition: Existence Relativization
Let $\mathcal{E}$ be a unary relation symbol. The *existence relativization* of a formula ϕ, denoted $\phi^{\mathcal{E}}$ is defined as follows:

i. If ϕ is atomic, $\phi^{\mathcal{E}} = \phi$

ii. $(\neg\phi)^{\mathcal{E}} = \neg(\phi^{\mathcal{E}})$

iii. For a binary connective $\circ$, $(\phi \circ \psi)^{\mathcal{E}} = \phi^{\mathcal{E}} \circ \psi^{\mathcal{E}}$

iv. $(\Box\phi)^{\mathcal{E}} = \Box\phi^{\mathcal{E}}$

v. $(\Diamond\phi)^{\mathcal{E}} = \Diamond\phi^{\mathcal{E}}$

vi. $(\forall x\phi)^{\mathcal{E}} = \forall x(\mathcal{E}(x) \supset \phi^{\mathcal{E}})$

vii. $(\mathcal{E}x\phi)^{\mathcal{E}} = \exists x(\mathcal{E}(x) \wedge \phi^{\mathcal{E}})$

► **Theorem 1.5.1.** *Let ϕ be a sentence not containing the symbol $\mathcal{E}$. Then, ϕ is valid in every varying domain model if and only if $\phi^{\mathcal{E}}$ is valid in every constant domain model.*

Proof Sketch. *Show that possessing a varying domain model $\mathcal{M}$ that makes ϕ invalid allows for the construction of a constant domain model which also makes $\phi^{\mathcal{E}}$ invalid. Now, show the reverse.*

1.6 Barcan and Converse Barcan Formulas

Definition: Barcan Formula
All formulas of the following forms are *Barcan formulas*:

$$\forall x(\Box\phi) \supset \Box\forall x\phi$$

$$\Diamond\mathcal{E}x\phi \supset \mathcal{E}x(\Diamond\phi)$$

Definition: Converse Barcan Formula
All formulas of the following forms are *Converse Barcan formulas*:

$$\Box\forall x\phi \supset \forall x(\Box\phi)$$

$$\mathcal{E}x(\Diamond\phi) \supset \Diamond\mathcal{E}x\phi$$

Note that in both cases above, each formula schema is the contrapositive of the other.

It is customary to speak of the Barcan and converse Barcan formulas as single entities; thus, saying the Barcan or converse Barcan formula is valid means that all of them are. Similarly, saying the Barcan or converse Barcan formula is not valid means that at least one of them isn't valid. It turns out, that neither the converse Barcan, nor Barcan formulas are valid in the largest model class under consideration; that is, in K.

Definition: Monotonic
The varying domain augmented frame $\langle g, \mathcal{R}, \mathcal{D} \rangle$ is *monotonic* provided that for every γ, $\delta \in g$, if $\gamma \mathcal{R} \delta$, then $\mathcal{D}(\gamma) \subseteq \mathcal{D}(\delta)$. A model is monotonic if and only if its frame is.

► **Theorem 1.6.1.**
A varying domain augmented frame $\mathcal{F}$ is monotonic if and only if the converse Barcan formula is frame valid on $\mathcal{F}$.

Definition: Anti-monotonic
The varying domain augmented frame $\langle g, \mathcal{R}, \mathcal{D} \rangle$ is *anti-monotonic* provided that for every γ, $\delta \in g$, if $\gamma \mathcal{R} \delta$, then $\mathcal{D}(\delta) \subseteq \mathcal{D}(\gamma)$. A model is anti-monotonic if and only if its frame is.

► **Theorem 1.6.2.**
A varying domain augmented frame $\mathcal{F}$ is anti-monotonic if and only if the Barcan formula is frame valid on $\mathcal{F}$.

Definition: Locally Constant Domain
The varying domain augmented frame $\langle g, \mathcal{R}, \mathcal{D} \rangle$ has *locally constant domain* provided that for every γ, $\delta \in g$, if $\gamma \mathcal{R} \delta$, then $\mathcal{D}(\gamma) = \mathcal{D}(\delta)$. A model has locally constant domain if and only if its frame does.

► **Theorem 1.6.3.**
A well-formed formula ϕ is valid in all locally constant domain models if and only if ϕ is valid in all constant domain models.

Bibliography

[1] Fitting and Mendelsohn. *First-Order Modal Logic*