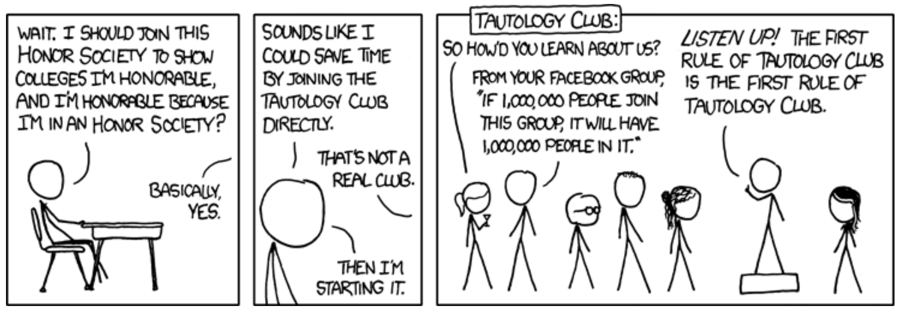
Worksheet 9: Derivatives and Limits

Russell Buehler
b.r@berkeley.edu



www.xkcd.com

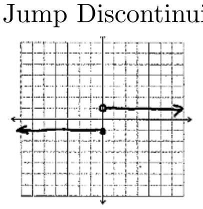
1. If $g(x) = x^4 - 2$, find $g'(1)$ using the definition of the derivative and use it to find the equation of the tangent line of $g(x)$ at $(1, -1)$.

$$\begin{aligned}
\lim_{h\to 0} \frac{g(x+h) - g(x)}{h} &= \lim_{h\to 0} \frac{(x+h)^4 - 2 - (x^4 - 2)}{h} \\
&= \lim_{h\to 0} \frac{(x^2 + 2xh + h^2)^2 - x^4}{h} \\
&= \lim_{h\to 0} \frac{x^4 + 2x^3h + x^2h^2 + 2x^3h + 4x^2h^2 + 2xh^3 + h^2x^2 + 2xh^3 + h^4 - x^4}{h} \\
&= \lim_{h\to 0} \frac{2x^3h + x^2h^2 + 2x^3h + 4x^2h^2 + 2xh^3 + h^2x^2 + 2xh^3 + h^4}{h} \\
&= \lim_{h\to 0} 2x^3 + x^2h + 2x^3 + 4x^2h + 2xh^2 + hx^2 + 2xh^2 + h^3 \\
&= 2x^3 + 2x^3 \\
&= 4x^3
\end{aligned}$$

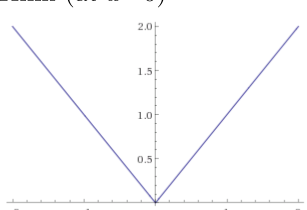
It follows that $g'(1) = 4$. We therefore have $y = 4x + b$ as the equation of our tangent line. Substituting $(1, -1)$, $-1 = 4(1) + b$, and thus $b = -5$. The equation of the tangent line of $g(x)$ at $(1, -1)$ is therefore $y = 4x - 5$.

2. List, with either an example graph or function and non-differentiable point, the ways in which a function can fail to be differentiable:

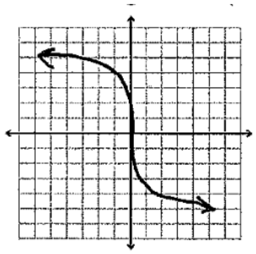
   (a) Jump Discontinuity (at x=0)

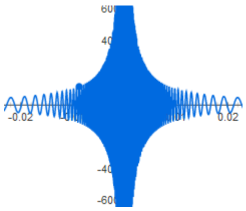
   (b) Kink (at x=0)

(c) Vertical Tangent Line (at x=0)



(d) Infinite Oscillation (at x=0)



3. (True or False) and why:

   (a) If a function is differentiable, then it is continuous.
   True; theorem from the book (pg. 158).

   (b) If a function is continuous, then it is differentiable.
   False; $f(x) = |x|$ is continuous, but not differentiable.

4. Give the physics interpretation for each of the following:

   (a) First Derivative – Velocity

   (b) Second Derivative – Acceleration

   (c) Third Derivative – Jerk

5. (True or False) and why.

   (a) If $\lim_{x\to 5} f(x) = 0$ and $\lim_{x\to 5} g(x) = 0$, then $\lim_{x\to 5} \frac{f(x)}{g(x)}$ does not exist.
   False; $f(x) = g(x) = x - 5$ is a counterexample.

   (b) If neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists, then $\lim_{x\to a} f(x) + g(x)$ does not exist.
   False; $f(x) = \frac{x+1}{x}, g(x) = \frac{-1}{x}$, and $a = 0$ is a counterexample.

   (c) If the limit $\lim_{x\to 6} f(x)g(x)$ exists, then the limit is $f(6)(g(6))$.
   False; let $f(x) = g(x)$ where $f(6) \neq \lim_{x\to 6} f(x)$ (move the value at 6 only).

   (d) If $p(x)$ is a polynomial, then the limit $\lim_{x\to 6} p(x)$ is $p(6)$.
   True; it's a direct result of the limit laws (see pg.101)

   (e) If $\lim_{x\to 0} f(x) = \infty$ and $\lim_{x\to 0} g(x) = \infty$, then $\lim_{x\to 0} f(x) - g(x) = 0$.
   False; $f(x) = \frac{2}{x}$, $g(x) = \frac{1}{x}$ is a counterexample.

6. Solve:

(a) $\lim_{x\to\infty} \frac{\sqrt{x^2-9}}{2x-6}$

$$\begin{aligned}
\lim_{x\to\infty} \frac{\sqrt{x^2-9}}{2x-6} &= \lim_{x\to\infty} \frac{\sqrt{x^2-9}}{2x-6}\left(\frac{\frac{1}{x}}{\frac{1}{x}}\right) \\
&= \lim_{x\to\infty} \frac{\sqrt{x^2-9}}{2x-6}\left(\frac{\frac{1}{x}}{\frac{1}{x}}\right) \\
&= \lim_{x\to\infty} \frac{\frac{\sqrt{x^2-9}}{x}}{\frac{2x-6}{x}} \\
&= \lim_{x\to\infty} \frac{\frac{\sqrt{x^2-9}}{\sqrt{x^2}}}{2-\frac{6}{x}} \\
&= \lim_{x\to\infty} \frac{\sqrt{\frac{x^2-9}{x^2}}}{2-\frac{6}{x}} \\
&= \lim_{x\to\infty} \frac{\sqrt{1-\frac{9}{x^2}}}{2-\frac{6}{x}} \\
&= \frac{\sqrt{1-0}}{2-0} \\
&= \frac{1}{2}
\end{aligned}$$

(b) $\lim_{x\to 1} e^{x^3-x}$

This limit doesn't actually exist...which I wasn't expecting when I chose it; apologies.

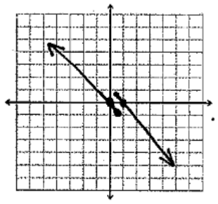
(c) $\lim_{x\to 3} \frac{\sqrt{x+6}-x}{x^3-3x^2}$

$$\begin{aligned}
\lim_{x\to 3} \frac{\sqrt{x+6}-x}{x^3-3x^2} &= \lim_{x\to 3} \frac{\sqrt{x+6}-x}{x^3-3x^2}\left(\frac{\sqrt{x+6}+x}{\sqrt{x+6}+x}\right) \\
&= \lim_{x\to 3} \frac{x+6-x^2}{(x^3-3x^2)(\sqrt{x+6}+x)} \\
&= \lim_{x\to 3} \frac{-(x-3)(x+2)}{(x^2(x-3))(\sqrt{x+6}+x)} \\
&= \lim_{x\to 3} \frac{-(x+2)}{(x^2)(\sqrt{x+6}+x)} \\
&= \frac{-5}{(9)(3+3)} \\
&= -\frac{5}{54}
\end{aligned}$$

(d) $\lim_{x\to\pi^-} \ln(\sin(x))$

Analyzing the limit conceptually, note that as we approach π from the left, $\sin(x)$ approaches 0 from the right. If $\sin(x)$ is approaching 0 from the right, then $\ln(\sin(x))$ is approaching $-\infty$ (recall the ln graph).

7. Sketch the graph of a function for which $f(0) = 0$, $f'(0) = -1$, $f(1) = 0$, and $f'(1) = -1$.



8. Write the general form for:

   (a) The Power Rule – $\frac{d}{dx}(x^n) = nx^{n-1}$ for n a real number

   (b) The Constant Multiple Rule – $\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$ for a constant c and f differentiable

   (c) The Sum Rule – if f, g differentiable, then $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$

   (d) The Difference Rule – if f, g differentiable, then $\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$

9. Find the first and second derivative of: $f(x) = 6x^{-\frac{8}{3}}$. Express them in both major notations.

   Applying the rules above, $f'(x) = \frac{df}{dx} = 6(-\frac{8}{3})x^{-\frac{11}{3}} = -16x^{-\frac{11}{3}}$. Taking the derivative again, $f''(x) = \frac{d^2f}{dx^2} = -16(-\frac{11}{3})x^{-\frac{14}{3}} = \frac{176}{3}x^{-\frac{14}{3}}$.

10. Find the first and second derivative of: $f(x) = e^x - 5$. Express them in both major notations.

    Applying the rules above plus some knowledge of e^x, $f'(x) = \frac{df}{dx} = e^x$. Taking the derivative again, $f''(x) = \frac{d^2f}{dx^2} = e^x$.