# Worksheet 8: To Infinity and Beyond!

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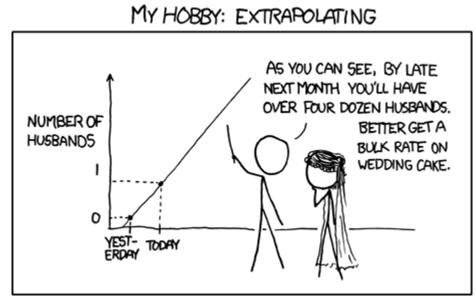



www.xkcd.com

1. Find the limit or show that it does not exist: $\lim_{x\to\infty} x^2 - x$

   Analyzing the form of the limit conceptually, we have $\infty - \infty$, an indeterminate form. Noting that $x^2 - x$ can be factored, we obtain $\lim_{x\to\infty} x^2 - x = \lim_{x\to\infty} x(x-1)$. A conceptual analysis now gives $\infty(\infty)$ as the form of the limit, and thus the limit is $\infty$.

2. Find the limit or show that it does not exist: $\lim_{x\to\infty} \frac{\sqrt{9x^6 - x}}{x^3+1}$

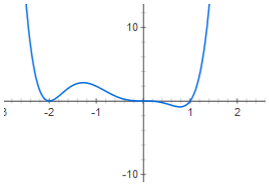
$$\begin{aligned}
\lim_{x\to\infty} \frac{\sqrt{9x^6 - x}}{x^3+1} &= \lim_{x\to\infty} \frac{\sqrt{9x^6 - x}}{x^3+1}\left(\frac{\frac{1}{x^3}}{\frac{1}{x^3}}\right)\\
&= \lim_{x\to\infty} \frac{\frac{\sqrt{9x^6-x}}{x^3}}{\frac{x^3+1}{x^3}}\\
&= \lim_{x\to\infty} \frac{\frac{\sqrt{9x^6-x}}{\sqrt{x^6}}}{1+\frac{1}{x^3}}\\
&= \lim_{x\to\infty} \frac{\sqrt{\frac{9x^6-x}{x^6}}}{1+\frac{1}{x^3}}\\
&= \lim_{x\to\infty} \frac{\sqrt{9+\frac{1}{x^5}}}{1+\frac{1}{x^3}}
\end{aligned}$$

   Noting that constant terms over positive powers of $x$ go to zero as $x \to \infty$,

$$\begin{aligned}
&= \frac{\sqrt{9+0}}{1+0}\\
&= 3
\end{aligned}$$

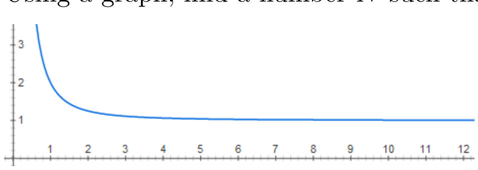
3. Find the limit as $x \to \infty$ and $x \to -\infty$, use this information as well as the $x$ and $y$ intercepts to give a rough sketch of the graph:

$$f(x) = x^3(x+2)^2(x-1)$$



   Note that $f$ has $x$-intercepts at 0, -2, and 1. Note further that $f$ has its only $y$-intercept at 0 (set $x = 0$). Noting that $f$ is positive for $x > 1$ and $x < -2$, we have all the information needed to make a rough sketch of $f$'s graph.

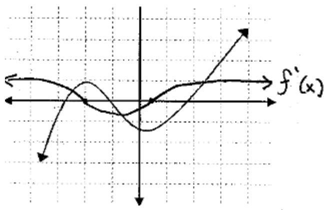
4. Using a graph, find a number $N$ such that if $x > N$, then $|\frac{x^2+1}{x^2} - 1| < \frac{1}{2}$.



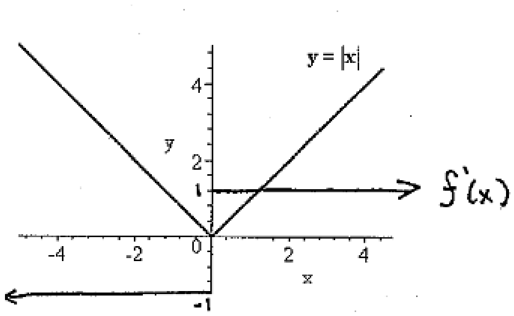
   Plugging in a few points shows the general pattern given in the graph to the left. In general, note that any $x$-value greater than or equal to 2 is clearly within the $(\frac{1}{2}, \frac{3}{2})$ range. It follows immediately that $N = 2$ or any greater value are satisfactory.

5. For each graph of a function $f(x)$ shown below; sketch the graph of its derivative.

(a)




(b)




6. Find the derivative of the function using the definition of derivative. State the domain of the function and the domain of its derivative.

(a) $f(x) = mx + b$

Applying the definition of the derivative,

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{m(x+h)+b-(mx+b)}{h}\\ &= \lim_{h\to 0}\frac{mx+mh+b-mx-b}{h}\\ &= \lim_{h\to 0}\frac{mh}{h}\\ &= \lim_{h\to 0} m\\ &= m\end{aligned}$$

Both $f(x)$ and its derivative have all real numbers as their domain.

(b) $f(t) = 5t - 9t^2$

Applying the definition of the derivative,

$$\begin{aligned}\lim_{h\to 0}\frac{f(t+h)-f(t)}{h} &= \lim_{h\to 0}\frac{5(t+h)-9(t+h)^2-(5t-9t^2)}{h}\\ &= \lim_{h\to 0}\frac{5t+5h-9(t^2+2th+h^2)-5t+9t^2}{h}\\ &= \lim_{h\to 0}\frac{5h-9t^2-18th-9h^2+9t^2}{h}\\ &= \lim_{h\to 0}\frac{5h-18th-9h^2}{h}\\ &= \lim_{h\to 0} 5-18t-9h\\ &= 5-18t\end{aligned}$$

Both $f$ and its derivative have all real numbers as their domain.

(c) $f(x) = x^4$

Applying the definition of the derivative,

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \frac{(x+h)^4 - x^4}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^4 - x^4}{h} \\
&= \lim_{h\to 0} \frac{(x+h)(x+h)(x+h)(x+h) - x^4}{h} \\
&= \lim_{h\to 0} \frac{(x^2+2xh+h^2)(x^2+2xh+h^2) - x^4}{h} \\
&= \lim_{h\to 0} \frac{(x^4+2x^3h+x^2h^2+2x^3h+4x^2h^2+2xh^3+x^2h^2+2xh^3+h^4) - x^4}{h} \\
&= \lim_{h\to 0} \frac{2x^3h+x^2h^2+2x^3h+4x^2h^2+2xh^3+x^2h^2+2xh^3+h^4}{h} \\
&= \lim_{h\to 0} 2x^3+x^2h+2x^3+4x^2h+2xh^2+x^2h+2xh^2+h^3 \\
&= 2x^3+2x^3 \\
&= 4x^3
\end{aligned}$$

Both $f$ and its derivative have all real numbers as their domain.