Worksheet 7: Continuity!

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1. Find the numbers at which f is discontinuous. At which of these numbers is f continuous from the right, from the left, or neither.

$$f(x) = \begin{cases} x+1 & : x \leq 1 \\ \frac{1}{x} & : 1 < x < 3 \\ \sqrt{x-3} & : x \geq 3 \end{cases}$$

f is discontinuous at 1 and 3 since $x+1 \neq \frac{1}{x}$ at 1 and $\frac{1}{x} \neq \sqrt{x-3}$ at 3. By definition, f is continuous from the left at 1 since the limit exists and is equal to $f(1)$ ($\lim\limits_{x \to 1^-} x+1 = 2 = f(1)$). Similarly for 3 from the right. f is not however continuous from the right at 1 or from the left at 3 since the one-sided limits are not equal to $f(1)$ and $f(3)$ respectively.

2. Use the intermediate value theorem to show that there is a root of $f(x) = -e^x + 3 - 2x$ in the interval $(0, 1)$.

Note first that $f(x)$ is continuous on $[0, 1]$ since it is an algebraic combination of functions which are continuous over this interval. Note further that $f(0) \neq f(1)$ and that $f(0) = 2 > 0 > f(1) = 1 - e$. It follows by the intermediate value theorem that $f(x) = 0$ somewhere on the interval $(0, 1)$.

3. Find a constant c such that

$$g(x) = \begin{cases} x^2 - c^2 & : x < 4 \\ cx + 20 & : x \geq 4 \end{cases}$$

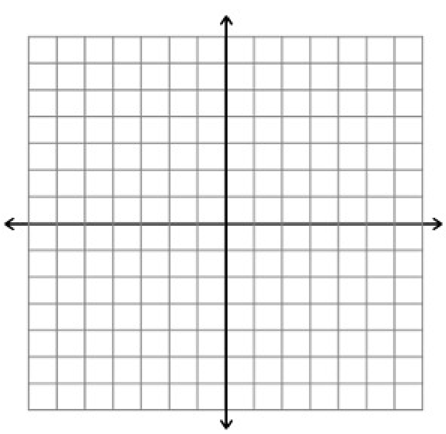
is continuous.

Note that $g(x)$ will be continuous if the two functions given are equal at 4. Thus,

$$\begin{aligned} 4^2 - c^2 &= 4c + 20 \\ 16 - c^2 &= \\ c^2 &= 4c + 4 \\ 0 &= c^2 + 4c + 4 \\ 0 &= (c+2)(c+2) \end{aligned}$$

And thus, $c = -2$.

4. Sketch a function $f(x)$ such that $\lim\limits_{x \to 3} f(x) = -\infty$, $\lim\limits_{x \to \infty} f(x) = 2$, $f(0) = 0$, and f is even.



5. Evaluate $\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$

$$\begin{aligned}
\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} &= \lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} \left(\frac{\frac{1}{x^2}}{\frac{1}{x^2}} \right) \\
&= \lim_{x \to \infty} \frac{3 - \frac{1}{x} - \frac{2}{x^2}}{5 + \frac{4}{x^2} + \frac{1}{x^2}} \\
&= \frac{3}{5}
\end{aligned}$$

6. Evaluate $\lim_{x \to \infty} \sqrt{x^2 - 3} - x$

$$\begin{aligned}
\lim_{x \to \infty} \sqrt{x^2 - 3} - x &= \lim_{x \to \infty} \sqrt{x^2 - 3} - x \left(\frac{\sqrt{x^2 - 3} + x}{\sqrt{x^2 - 3} + x} \right) \\
&= \lim_{x \to \infty} \frac{x^2 - 3 - x^2}{\sqrt{x^2 - 3} + x} \\
&= \lim_{x \to \infty} \frac{-3}{\sqrt{x^2 - 3} + x} \\
&= 0
\end{aligned}$$