Worksheet 6: $\epsilon - \delta$ Limits and More!

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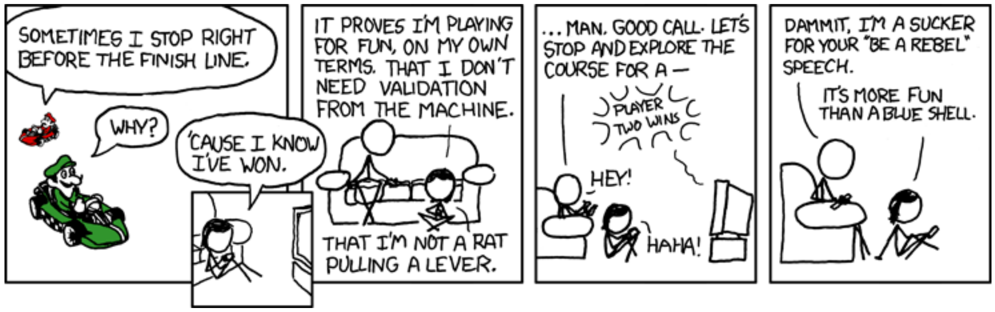

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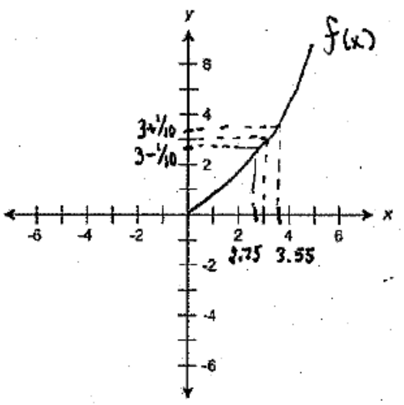
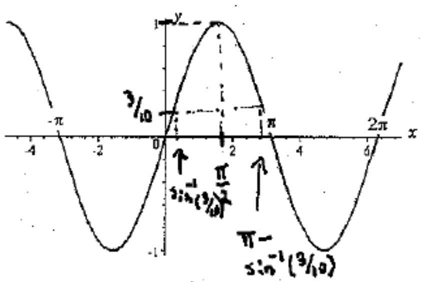
1. Let $\epsilon = \frac{1}{10}$. Use the graph below to find a δ that satisfies $|x-3| < \delta \Rightarrow |f(x) - 3| < \epsilon$.



Note first that the ϵ-neighborhood we need to pick a δ for is show on the y-axis with the corresponding x-values given. Since we need to pick a δ-value such that all x-values in $(3-\delta, 3+\delta)$ end up in the ϵ-neighborhood given, we may simply take $\delta = 3 - 2.75 = .25$, the smaller of the two intervals on either side of 3 on the x-axis. Because of this, δ must be less than or equal to .25 (any smaller, positive value is acceptable).

2. Let $\epsilon = \frac{7}{10}$. Find a δ that satisfies $|x - \frac{\pi}{2}| < \delta \Rightarrow |f(x) - 1| < \epsilon$ using $f(x) = \sin(x)$ and the graph below.



We begin by drawing in the ϵ-neighborhood we need to pick a δ for, as show on the y-axis with the corresponding x-values given. Note that having an ϵ-interval that includes y-values which the function cannot reach isn't a problem. Since we need to pick a δ-value such that all x-values in $(\frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta)$ end up in the ϵ-neighborhood given, we may simply take $\delta = \frac{\pi}{2} - \sin^{-1}(\frac{3}{10})$, the distance to either of the x-values given being equal (again, any smaller, positive value is also acceptable).

3. Let $f(x) = 12x$. Show that $\lim_{x \to 12} f(x) = 144$ using the $\epsilon - \delta$ definition of limit.

Scratch Work
Filling values into the $\epsilon - \delta$ definition, I want:
$|x - 12| < \delta \Rightarrow |12x - 144| < \epsilon$.
Fix ϵ and consider $|12x - 144| < \epsilon$; in particular, note that I can manipulate this to look like the left side of the $\Rightarrow$ statement above:

$$|12x - 144| < \epsilon$$
$$12|x - 12| < \epsilon$$
$$|x - 12| < \frac{\epsilon}{12}$$

Since the left side of the statement above is the same as the δ portion of the $\Rightarrow$ statement and the right is a constant (remember, ϵ is some fixed value right now), I now know what I should set δ equal to and can write the actual solution (right).

Actual Solution
Let $\epsilon > 0$, and set $\delta = \frac{\epsilon}{12}$. Consider then,

$$|x - 12| < \delta$$
$$|x - 12| < \frac{\epsilon}{12}$$
$$(12)|x - 12| < \epsilon$$
$$|12x - 144| < \epsilon$$

It follows immediately that if $|x - 12| < \delta$, then $|12x - 144| < \epsilon$. Furthermore, notice that the ϵ used was arbitrary, and so–by definition– $\lim_{x \to 12} f(x) = 144$.

4. Let $f(x) = x^2$. Show that $\lim_{x\to 0} f(x) = 0$ using the $\epsilon - \delta$ definition of limit.

Scratch Work
Filling values into the $\epsilon - \delta$ definition, I want:
$|x - 0| < \delta \Rightarrow |x^2 - 0| < \epsilon$.
Fix ϵ and consider $|x^2 - 0| < \epsilon$; in particular, note that I can manipulate this to look like the left side of the $\Rightarrow$ statement above:

$$|x^2| < \epsilon$$
$$x^2 < \epsilon$$
$$-\sqrt{\epsilon} < x < \sqrt{\epsilon}$$
$$|x| < \sqrt{\epsilon}$$

Note that since only real numbers are possible the first step above (simply dropping the absolute value) is justified. Since the left side of the statement above is the same as the δ portion of the $\Rightarrow$ statement and the right is a constant (remember, ϵ is some fixed value right now), I now know what I should set δ equal to and can write the actual solution (right).

Actual Solution
Let $\epsilon > 0$, and set $\delta = \sqrt{\epsilon}$. Consider then,

$$|x - 0| < \delta$$
$$|x - 0| < \sqrt{\epsilon}$$
$$-\sqrt{\epsilon} < x < \sqrt{\epsilon}$$
$$x^2 < \epsilon$$
$$|x^2 - 0| < \epsilon$$

It follows immediately that if $|x - 0| < \delta$, then $|x^2 - 0| < \epsilon$. Furthermore, notice that the ϵ used was arbitrary, and so–by definition–$\lim_{x\to 0} f(x) = 0$.