Worksheet 5: Not PreCalc!

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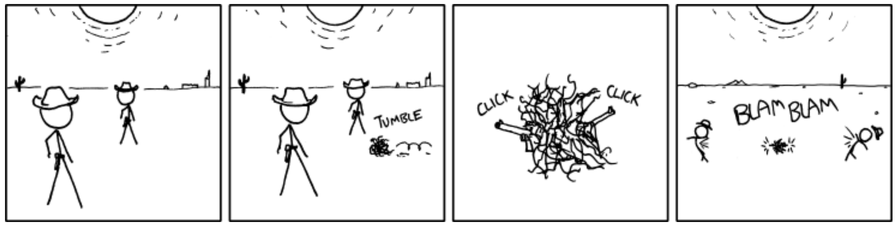

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1. Determine the infinite limit: $\lim_{x \to -3^-} \dfrac{x+2}{x+3}$.

2. Evaluate the limit: $\lim_{t \to 2} \dfrac{t^2 - 2}{t^3 - 3t + 5}$.

3. Evaluate the limit, if it exists:

   (a) $\lim_{t \to 1} \dfrac{t^4 - 1}{t^3 - 1}$.

   (b) $\lim_{t \to 0} \dfrac{1}{t\sqrt{1+t}} - \dfrac{1}{t}$.

   (c) $\lim_{x \to -6} \dfrac{2x + 12}{|x + 6|}$.

4. The formal definition of the limit is:

$$\lim_{x \to a} f(x) = L \text{ if and only if}$$

$$\forall \epsilon > 0,\ \exists \delta \text{ such that if } 0 < |x - a| < \delta, \text{ then } |f(x) - L| < \epsilon$$

   Explain the second line of the definition in simple English.

5. Let $f(x) = x^2 - 1$. Prove $\lim_{x \to 1} f(x) = 0$.

6. **A fun problem that you will not be held accountable for–at all–but that destroyed mathematician's careers in the late 1800s and early 1900s.**

   (a) If you recall from last week, we say that a function f is bijective if it is both one-to-one (injective) and onto (surjective). Bijective functions, then, are those functions which associate every x-value with its own y-value and leave no y-values without an x-value. Mathematicians sometimes make the claim that the existence of a bijective function between two sets means the sets have the same number of elements; explain.

   (b) Georg Cantor, a mathematician, applied the ideas above to infinite sets. Following him, is the set of all positive odd numbers the same 'size' as the set of all positive even numbers?

   (c) Let $\mathbb{N}$ denote the set of natural numbers $(0, 1, 2, 3, 4, \ldots)$ and $\mathbb{R}$ denote the set of real numbers. Assume that $\mathbb{N}$ is, in fact, the same size as $\mathbb{R}$ (both are, after all, infinitely large); it follows that $\mathbb{N}$ is at least as large as the interval $(0, 1)$. Taking our bijection between $\mathbb{N}$ and $\mathbb{R}$, then, we should be able to create a table as below (filling in random decimal expansions if a natural number was mapped to something not in $(0, 1)$ in the original bijection),

$$\begin{aligned} f(0) &= .123456789\ldots \\ f(1) &= .234567891\ldots \\ f(2) &= .345678912\ldots \\ f(3) &= .456789123\ldots \\ f(4) &= .567891234\ldots \\ \vdots \quad & \vdots \quad \vdots \end{aligned}$$

   wherein a decimal expansion for every value in $(0, 1)$ appears at least once on the right side. Can you find a number in the interval $(0, 1)$ that isn't in the table? Hint: Describe a rule that will give such a number.

   (d) What can you conclude about the relative 'sizes' of $\mathbb{N}$ and the interval $(0, 1)$? $\mathbb{N}$ and $\mathbb{R}$?