Worksheet 31: Volume

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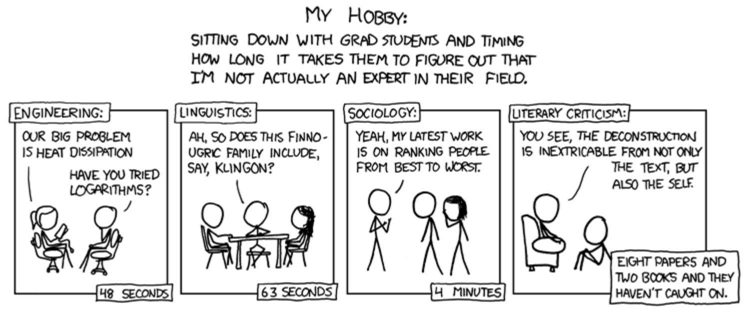


www.xkcd.com

1. If I want to find the area over $[a, b]$ bounded above and below by $f(x)$ and $g(x)$ respectively where both are continuous over the interval and $f(x) > g(x) > 0$ over the interval, what would I use for an integral?

   This is simply the area below $f(x)$ minus the area below $g(x)$; noting that integrals play nicely with subtraction,

$$\int_a^b (f(x) - g(x))dx$$

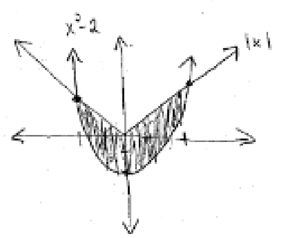
2. If I want to find the area over $[a, b]$ bounded above and below by $f(x)$ and $g(x)$ respectively where both are continuous over the interval, what would I use for an integral? How would I solve for the integral?

   Since we don't have any idea how $f(x)$ relates to $g(x)$,

$$\int_a^b |f(x) - g(x)|dx$$

3. Sketch the region enclosed by the given curves and find its area:

   (a) $y = |x|$, $y = x^2 - 2$



Looking at the graph, it's natural to take the integral with respect to the x-axis and break it at $x = 0$ to avoid dealing with the absolute value. Since $y = |x|$ is always above $y = x^2 - 2$, we need only find the intersections shown.

Breaking the absolute value,

$$\begin{aligned} -x &= x^2 - 2 \\ x^2 + x - 2 &= 0 \\ (x-1)(x+2) &= 0 \\ x &= 1, -2 \end{aligned}$$

Noting this is the $-x$ side, $x = 1$ isn't a viable option, so $x = -2$ is the intersection. Similarly,

$$\begin{aligned} x &= x^2 - 2 \\ x^2 - x - 2 &= 0 \\ (x+1)(x-2) &= 0 \\ x &= -1, 2 \end{aligned}$$

Noting this is the x side, $x = -1$ isn't a viable option, so $x = 2$ is the other intersection. Thus, the relevant integral is

$$\int_{-2}^{0}[-x-(x^2-2)]dx + \int_{0}^{2}[x-(x^2-2)]dx$$

Noting that the areas on each side of 0 are the same, we have

$$2\int_0^2 [x-(x^2-2)]dx$$

$$2\int_0^2 [-x^2+x+2]dx$$

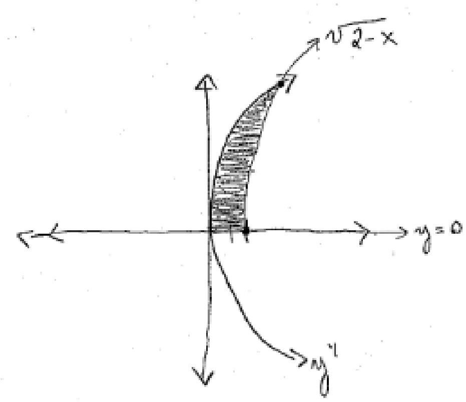
$$2\Big(-\frac{1}{3}x^3+\frac{1}{2}x^2+2x\Big]_0^2\Big)$$

$$2\Big(-\frac{1}{3}2^3+\frac{1}{2}2^2+4\Big)$$

$$-\frac{2^4}{3}+2^2+2^3$$

$$\frac{20}{3}$$

(b) $x=y^4, y=\sqrt{2-x}, y=0$



Looking at the graph, it's natural to take the integral with respect to the y-axis. Since $y=\sqrt{2-x}$ is always 'above' $x=y^4$, we need only find the intersection shown (the other bound being 0).

Setting the two equal,

$$\begin{aligned} y &= \sqrt{2-x} \\ y^2 &= 2-x \\ &= 2-y^4 \\ y^4+y^2-2 &= 0 \\ (y^2+2)(y^2-1) &= 0 \\ y &= \pm 1, \pm\sqrt{-2} \end{aligned}$$

Noting that the intersection is a positive y value in the real plane, $y=1$ is the desired intersection. Solving for y,

$$\begin{aligned} y &= \sqrt{2-x} \\ y^2 &= 2-x \\ x &= 2-y^2 \end{aligned}$$

Thus, the relevant integral is

$$\int_0^1 [2-y^2-y^4]dy$$

$$2y - \frac{1}{3}y^3 - \frac{1}{5}y^5\Big]_0^1$$

$$2 - \frac{1}{3} - \frac{1}{5}$$

$$\frac{22}{15}$$

4. What formulas represent the volume of a solid of revolution?

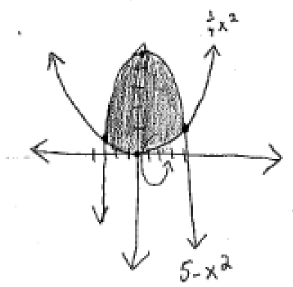
$$\int_a^b A(x)\,dx$$

$$\int_a^b A(y)\,dy$$

where $[a, b]$ is the interval on the x/y axis enclosing the solid and $A(x)/A(y)$ gives the area of a vertical/horizontal slice at x/y in the interval $[a, b]$.

5. Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

   (a) $y = \frac{1}{4}x^2, y = 5 - x^2$; about the x-axis

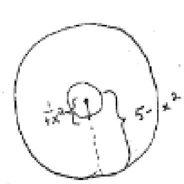


Looking at the graph, it's natural to take the integral with respect to the x-axis. We start by finding the interval over which we will integrate:

Setting the two equations equal,

$$\begin{aligned}\frac{1}{4}x^2 &= 5 - x^2\\ x^2 &= 20 - 4x^2\\ 5x^2 &= 20\\ x^2 &= 4\\ x &= \pm 2\end{aligned}$$

Next, we need an expression for $A(x)$, the area at any x-slice of the solid. Drawing an arbitrary slice gives,



$A(x)$ is thus the area of the larger circle minus the area of the smaller, $A(x) = \pi(5 - x^2)^2 - \pi(\frac{1}{4}x^2)^2$. The desired integral is thus,

$$\int_{-2}^{2} [\pi(5 - x^2)^2 - \pi(\frac{1}{4}x^2)^2]\,dx$$

$$\int_{-2}^{2} \pi[25 - 10x^2 + x^4 - \frac{1}{16}x^4]\,dx$$

$$\pi \int_{-2}^{2} [25 - 10x^2 + \frac{15}{16}x^4]\, dx$$

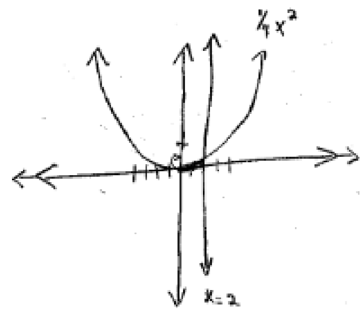
$$\pi \left(25x - \frac{10}{3}x^3 + \frac{15}{80}x^5 \Big]_{-2}^{2} \right)$$

$$2\pi \left(25(2) - \frac{10}{3}2^3 + \frac{15}{80}2^5 \right)$$

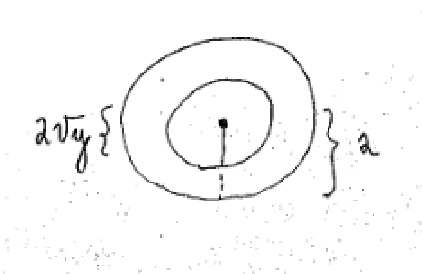
$$2\pi \left(50 - \frac{10}{3}2^3 + 6 \right)$$

$$\frac{176\pi}{3}$$

(b) $y = \frac{1}{4}x^2, x = 2, y = 0$; about the y-axis



Looking at the graph, it's natural to take the integral with respect to the y-axis (an x-slice is not a rectangle...). We start by noting that the interval over which we will integrate is [-2,2]. Next, we need an expression for $A(x)$, the area at any y-slice of the solid. Drawing an arbitrary slice gives,



$A(y)$ is thus the area of the larger circle minus the smaller, $A(y) = \pi(2)^2 - \pi(2\sqrt{y})^2 = \pi(4 - 4y)$. The desired integral is thus,

$$\int_{-2}^{2} \pi(4 - 4y)\, dy$$

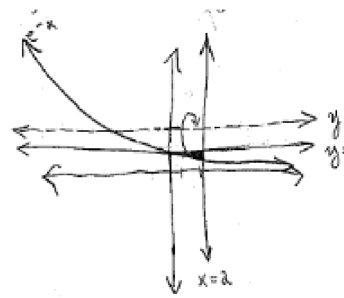
$$\pi \int_{-2}^{2} (4 - 4y)\, dy$$

$$\pi \left[4y - 2y^2 \right]_{-2}^{2}$$

$$\pi \left[8 - 8 - (-8 - 8) \right]$$

$$16\pi$$

(c) $y = e^{-x}$, $y = 1, x = 2$; about $y = 2$



Looking at the graph, it's natural to take the integral with respect to the x-axis. We start by noting that the interval over which we will integrate is [0,2]. Next, we need an expression for $A(x)$, the area at any x-slice of the solid. Drawing an arbitrary slice gives,

$A(x)$ is thus the area of the larger circle minus the smaller, $A(x) = \pi(2 - e^{-x})^2 - \pi(1)^2 = \pi(2 - e^{-x})^2 - \pi$. The desired integral is thus,

$$\int_0^2 [\pi(2 - e^{-x})^2 - \pi]\, dx$$

$$\pi \int_0^2 [4 - 4e^{-x} + e^{-2x} - 1]\, dx$$

$$\pi \int_0^2 [3 - 4e^{-x} + e^{-2x}]\, dx$$

$$\pi \left[3x + 4e^{-x} - \frac{1}{2}e^{-2x}\right]_0^2$$

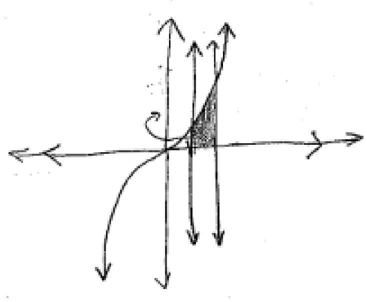
$$\pi \left[6 + 4e^{-2} - \frac{1}{2}e^{-4} - (4 - \frac{1}{2})\right]$$

$$\pi \left[\frac{5}{2} + 4e^{-2} - \frac{1}{2}e^{-4}\right]$$

$$\frac{\pi(5e^4 + 8e^2 - 1)}{2e^4}$$

6. Use the shell method to find the volume of the solid obtained by rotating the region bounded by the given curves about the y-axis.

   (a) $y = x^3$, $y = 0$, $x = 1$, $x = 2$



Looking at the graph, notice that the interval over which we will integrate is [1,2]. Since $f(x) = x^3$ is already given, and we wish to find the volume of a solid rotated around the y-axis, we meet all the conditions for the shell method and can simply plug in our values to get the necessary integral:

$$\int_1^2 (2\pi x)(x^3)\, dx$$

$$\int_1^2 2\pi x^4 \; dx$$

$$\frac{2\pi}{5}x^5\Big]_1^2$$

$$\frac{2\pi}{5}2^5 - \frac{2\pi}{5}1^5$$

$$\frac{2\pi}{5}(2^5 - 1)$$