Worksheet 30: Substitutions

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1. Evaluate:

   (a) $\int \sqrt{x^2-1}x^5\ dx$

   Let $u = x^2 - 1$, and thus $du = 2x\ dx$. Substituting in,

$$\begin{aligned}\int \sqrt{x^2-1}x^5\ dx &= \int \sqrt{u}\ \frac{1}{2}du\ (x^4)\\ &= \frac{1}{2}\int \sqrt{u}\ du\ (x^4)\end{aligned}$$

   Noting that $x^2 = u + 1$ by above,

$$\begin{aligned}&= \frac{1}{2}\int \sqrt{u}\ du\ (u+1)^2\\ &= \frac{1}{2}\int \sqrt{u}\ (u^2+2u+1)\ du\\ &= \frac{1}{2}\int (u^{\frac{5}{2}} + 2u^{\frac{3}{2}} + u^{\frac{1}{2}})\ du\\ &= \frac{1}{2}[\frac{2}{7}u^{\frac{7}{2}} + \frac{4}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}] + C\\ &= \frac{1}{7}u^{\frac{7}{2}} + \frac{2}{5}u^{\frac{5}{2}} + \frac{1}{3}u^{\frac{3}{2}} + C\end{aligned}$$

   And so, substituting back in:

$$= \frac{1}{7}(x^2-1)^{\frac{7}{2}} + \frac{2}{5}(x^2-1)^{\frac{5}{2}} + \frac{1}{3}(x^2-1)^{\frac{3}{2}} + C$$

   (b) $\int \frac{z^2}{z^3+1}\ dz$

   Let $u = z^3 + 1$, and thus $du = 3z^2\ dz$. Substituting in,

$$\begin{aligned}\int \frac{z^2}{z^3+1}\ dz &= \int \frac{\frac{1}{3}du}{u}\\ &= \frac{1}{3}\int \frac{du}{u}\\ &= \frac{1}{3}\ln(u) + C\end{aligned}$$

   And so, substituting back in:

$$= \frac{1}{3}\ln(z^3+1) + C$$

(c) $\int \frac{\sin(x)}{1+\cos^2(x)}\ dx$

Let $u = 1 + \cos^2(x)$, and thus $du = -2\sin(x)\ dx$. Substituting in,

$$\int \frac{\sin(x)}{1+\cos^2(x)}\ dx = \int \frac{-\frac{1}{2}du}{u}$$
$$= -\frac{1}{2}\int \frac{du}{u}$$
$$= -\frac{1}{2}\ln(u) + C$$

And so, substituting back in:

$$= -\frac{1}{2}\ln(1+\cos^2(x)) + C$$

(d) $\int \sqrt{x^2+1}x^3\ dx$

Let $u = x^2 + 1$, and thus $du = 2x\ dx$. Substituting in,

$$\int \sqrt{x^2+1}(x^3)\ dx = \int \sqrt{u}(\frac{1}{2})du(x^2)$$
$$= \frac{1}{2}\int \sqrt{u}\ du\ (x^2)$$

By above, $u = x^2 + 1$, and so $x^2 = u - 1$:

$$= \frac{1}{2}\int \sqrt{u}\ du\ (u-1)$$
$$= \frac{1}{2}\int [u^{\frac{3}{2}} - \sqrt{u}]\ du$$
$$= \frac{1}{2}[\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}] + C$$
$$= \frac{1}{5}u^{\frac{5}{2}} - \frac{1}{3}u^{\frac{3}{2}} + C$$

And so, substituting back in:

$$= \frac{1}{5}(x^2+1)^{\frac{5}{2}} - \frac{1}{3}(x^2+1)^{\frac{3}{2}} + C$$

2. Evaluate:

(a) $\int_1^e \frac{\ln(x)}{x}\ dx$

Let $u = \ln(x)$, and thus $du = \frac{1}{x}\ dx$. Substituting in,

$$\int_1^e \frac{\ln(x)}{x}\ dx = \int_0^1 u\ du$$
$$= \frac{1}{2}u^2\Big]_0^1$$
$$= \frac{1}{2}$$

(b) $\int_0^1 xe^{-x^2}\ dx$

Let $u = -x^2$, and thus $du = -2x\ dx$. Substituting in,

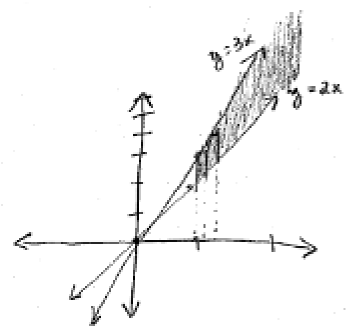
$$\int_0^1 xe^{-x^2}\ dx = \int_0^{-1} -\frac{1}{2}e^u\ du$$
$$= -\frac{1}{2}e^u\Big]_0^{-1}$$
$$= -\frac{1}{2}e^{-1} + \frac{1}{2}$$

(c) $\int_0^a x\sqrt{a^2 - x^2}\ dx$

Let $u = a^2 - x^2$, and thus $du = -2x\ dx$. Substituting in,

$$\int_0^a x\sqrt{a^2 - x^2}\ dx = \int_{a^2}^0 -\frac{1}{2}\sqrt{u}\ du$$
$$= -\frac{1}{2}\int_{a^2}^0 \sqrt{u}\ du$$
$$= -\frac{1}{3}u^{\frac{3}{2}}\Big]_{a^2}^0$$
$$= \frac{1}{3}a^3$$

3. Sketch the lines $y = 2x$ and $y = 3x$. Construct a series expression that gives exactly the area between the two over the interval $[0, 2]$.



Note that we may simply divide the interval $[0, 2]$ into n rectangles as usual, then subtract the (right-endpoint) rectangle for $y = 2x$ from the (right-endpoint) rectangle for $y = 3x$. This gives,

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2-0}{n}[3(\frac{2-0}{n}i) - 2(\frac{2-0}{n}i)] = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}[3(\frac{2}{n}i) - 2(\frac{2}{n}i)]$$

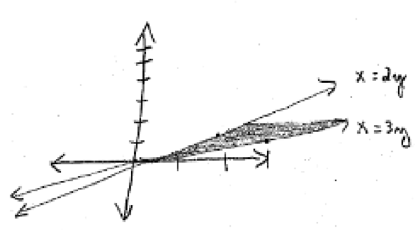
4. How might you express the area from (3) as an integral?

Looking at the definition of the integral, the above corresponds to

$$\int_0^2 (3x - 2x)\ dx$$

5. What if the lines were $x = 3y$, $x = 2y$, and I wanted the area between the two horizontally ([0,2] on the y-axis)?

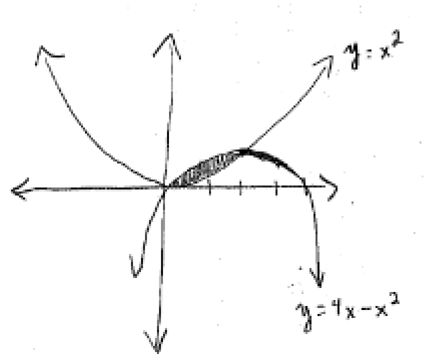
We simply do the same thing only with respect to the y-axis:



$$\int_0^2 (3y - 2y)\, dy$$

6. Sketch the region enclosed by the two curves and find its area: $y = x^2$, $y = 4x - x^2$

   Note first that $y = 4x - x^2 = x(4 - x)$. This graph is thus an upside down parabola which crosses the x-axis at 0 and 4. Drawing both graphs gives:



Working with respect to the x-axis (we could do the y-axis, if we desired), the top function is always $y = x^2$ and the interval is $[0, 2]$. Thus the desired area is

$$\int_0^2 x^2 - (4x - x^2)\, dx$$

$$\int_0^2 x^2 - 4x + x^2\, dx$$

$$\int_0^2 2x^2 - 4x\, dx$$

$$\frac{2}{3}x^3 - 2x^2\Big]_0^2 = \frac{2^4}{3} - 2^3$$