Worksheet 29: The Fundamental Thm. of Calculus

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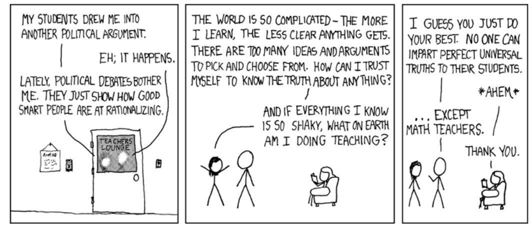

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1. The fundamental theorem of calculus has one assumption and two parts (see page. 393 if you don't remember).
   (a) What is the assumption?
   $f(x)$ is continuous over $[a, b]$
   (b) What are the two conclusions?
   - If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$
   - $\int_a^b f(x)\,dx = F(b) - F(a)$ where F is any anti-derivative of f

2. What, conceptually, is a function of the form $g(x) = \int_a^x f(t)\,dt$? How is x constrained?
   $g(x)$ gives the area under f from a to x; x is required to be in the interval $[a, b]$.

3. Find the derivative of the following:
   (a) $g(x) = \int_3^x e^{t^2-t}\,dt$
   Using the first part of the fundamental theorem of calculus,
$$g'(x) = f(x) = e^{x^2-x}$$
   (b) $g(r) = \int_3^r \sqrt{x^2+4}\,dx$
   Using the first part of the fundamental theorem of calculus,
$$g'(r) = f(r) = \sqrt{r^2+4}$$
   (c) $G(x) = \int_x^1 \cos(\sqrt{t})\,dt$
   Note that $G(x) = \int_x^1 \cos(\sqrt{t})\,dt = -\int_1^x \cos(\sqrt{t})\,dt$. Thus, using the first part of the fundamental theorem of calculus,
$$G'(x) = -f(x) = -\cos(\sqrt{x})$$
   (d) $y = \int_0^{x^4} \cos^2(\theta)\,d\theta$
   Note that the first part of the fundamental theorem of calculus only allows for the derivative with respect to the upper limit (assuming the lower is constant). In this case, however, the upper limit isn't just x, but rather x^4. We want, as earlier, to find
$$\frac{d}{dx}\left[\int_0^{x^4} \cos^2(\theta)\,d\theta\right]$$
   But the fundamental theorem applies to
$$\frac{d}{dx^4}\left[\int_0^{x^4} \cos^2(\theta)\,d\theta\right]$$
   The solution is to notice that $\frac{d}{dx} = \frac{dx^4}{dx}\frac{d}{dx^4}$. Thus,
$$\frac{d}{dx}\left[\int_0^{x^4} \cos^2(\theta)\,d\theta\right] = \frac{dx^4}{dx}\frac{d}{dx^4}\left[\int_0^{x^4} \cos^2(\theta)\,d\theta\right]$$
   And so, by the fundamental theorem,
$$\frac{dx^4}{dx}\cos^2(x^4)$$
$$4x^3\cos^2(x^4)$$

4. Evaluate the integral
   (a) $\int_{-1}^1 x^{100}\,dx$
   Using the fundamental theorem,
$$\int_{-1}^1 x^{100}\,dx = \frac{1}{101}x^{101}\Big]_{-1}^1 = \frac{1}{101}(1)^{101} - \frac{1}{101}(-1)^{101} = \frac{2}{101}$$

(b) $\int_{-5}^{5} e\ dx$
Using the fundamental theorem,

$$\int_{-5}^{5} e\ dx = ex\Big]_{-5}^{5} = 5e - (-5e) = 10e$$

(c) $\int_0^{\frac{\pi}{4}} \sec(\theta)\tan(\theta)\ d\theta$
Using the fundamental theorem,

$$\int_0^{\frac{\pi}{4}} \sec(\theta)\tan(\theta)\ d\theta = \sec(\theta)\Big]_0^{\frac{\pi}{4}} = \sqrt{2} - 1$$

5. What's wrong?

$$\int_{-1}^{2} \frac{4}{x^3}\ dx = \frac{-2}{x^2}\Big]_{-1}^{2} = \frac{3}{2}$$

$f(x)$ isn't continuous over $[-1, 2]$ (look at 0), so the fundamental theorem doesn't apply!

6. What's the difference between $\int_0^1 (x^3 + x + 1)\ dx$ and $\int x^3 + x + 1$? Find each.

The definite integral ($\int_0^1 (x^3 + x + 1)\ dx$) represents a number whereas the indefinite integral ($\int x^3 + x + 1$) represents a class of functions, in particular the anti-derivatives of $x^3 + x + 1$. Taking the anti-derivative, $\int x^3 + x + 1 = \frac{1}{4}x^4 + \frac{1}{2}x^2 + x + C$. By the fundamental theorem,

$$\int_0^1 (x^3 + x + 1)\ dx = \frac{1}{4}x^4 + \frac{1}{2}x^2 + x\Big]_0^1 = \frac{1}{4} + \frac{1}{2} + 1 - 0 = \frac{7}{4}$$

7. Verify the following:

$$\int \cos^3(x)\ dx = \sin(x) - \frac{1}{3}\sin^3(x) + C$$

Taking the derivative of the right side,

$$\cos(x) - \frac{1}{3}(3)\sin^2(x)(\cos(x))$$
$$\cos(x) - \sin^2(x)(\cos(x))$$

Using $\sin^2(x) + \cos^2(x) = 1$,

$$\cos(x) - (1 - \cos^2(x))(\cos(x))$$
$$\cos(x) - \cos(x) + \cos^3(x)$$
$$\cos^3(x)$$

8. Find the general indefinite integral:

(a) $\int 2^x\ dx$
Noting that $\frac{d}{dx}[a^x] = a^x \ln(a)$,

$$\int 2^x\ dx = \frac{1}{\ln(2)}2^x$$

(b) $\int \sec(t)(\sec(t) + \tan(t))\ dt$
Distributing,

$$\int \sec(t)(\sec(t) + \tan(t))\ dt = \int [\sec^2(t) + \sec(t)\tan(t)]\ dt = \tan(t) + \sec(t)$$

9. Evaluate the integral:

(a) $\int_0^3 (1 + 6w^2 - 10w^4)dw$
Using the fundamental theorem,

$$\int_0^3 (1 + 6w^2 - 10w^4)dw = w + 2w^3 - 2w^5\Big]_0^3 = 3 + 54 - 486 = 429$$

(b) $\int_0^1 (5x - 5^x)dx$
Using the fundamental theorem,

$$\int_0^1 (5x - 5^x)dx = \frac{5}{2}x^2 - \frac{1}{\ln(5)}5^x\Big]_0^1 = \frac{5}{2} - \frac{5}{\ln(5)} + \frac{1}{\ln(5)} = \frac{5}{2} - \frac{4}{\ln(5)}$$