# Worksheet 28: Definite Integrals

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1. Determine two different regions whose area is equal to the limit $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}(5+\frac{2i}{n})^{10}$

   If we interpret the function under consideration as $f(x) = x^{10}$, we obtain $[5, 7]$ as our interval. If we take the function to be $f(x) = (5+x)^{10}$, $[0, 2]$ is the corresponding interval.

2. Give a summation for the area of the function $f(x)$ over the interval $[a, b]$.

   Using right endpoints,

$$\sum_{i=1}^{n} \frac{b-a}{n} f(a + \frac{b-a}{n}(i))$$

3. Assuming that $\int_a^b f(x)dx$ exists, how does it relate to the solution in (2)?

$$\int_a^b f(x)dx = \lim_{n\to\infty} \text{ of the summation in (4)}$$

4. Express the following as an integral:

$$\lim_{n\to\infty} \sum_{i=1}^{n} [(a + \frac{(b-a)i}{n})^3 + (a + \frac{(b-a)i}{n}) \sin(a + \frac{(b-a)i}{n})](\frac{(b-a)}{n})$$

$$\int_a^b [x^3 + x\sin(x)]dx$$

5. What is the value of $\int_b^a f(x)dx$ (in terms of $\int_a^b f(x)dx$)? Why?

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

   Invoking problems 3 and 4,

$$\begin{aligned}
\int_a^b f(x)dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{b-a}{n} f(a + \frac{b-a}{n}(i)) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} -\frac{a-b}{n} f(a + \frac{b-a}{n}(i)) \\
&= -\lim_{n\to\infty} \sum_{i=1}^{n} \frac{a-b}{n} f(a + \frac{b-a}{n}(i)) \\
&= -\int_b^a f(x)dx
\end{aligned}$$

6. What is the value of $\int_a^a f(x)dx$? Why?

   Invoking problems 3 and 4,

$$\begin{aligned}
\int_a^a f(x)dx &= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{a-a}{n}f(a+\frac{a-a}{n}(i)) \\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{0}{n}f(a+\frac{0}{n}(i)) \\
&= \lim_{n\to\infty} 0 \\
&= 0
\end{aligned}$$

7. Find an expression for the value of $\int_a^b c\,dx$ where $c$ is any constant.
   Invoking problems 3 and 4,

$$\begin{aligned}
\int_a^b c\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{b-a}{n}\,c \\
&= \lim_{n\to\infty} n[\frac{b-a}{n}]\,c \\
&= \lim_{n\to\infty}(b-a)\,c \\
&= (b-a)\,c
\end{aligned}$$

8. ($\star$) Use a Riemann sum representation with right endpoints to find a value for $\int_2^5(4-2x)dx$
   Invoking problems 3 and 4,

$$\begin{aligned}
\int_2^5(4-2x)dx &= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{5-2}{n}\,[4-2(2+\frac{i(5-2)}{n})] \\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{12}{n}-\frac{12}{n}-\frac{18i}{n^2} \\
&= \lim_{n\to\infty}-\sum_{i=1}^{n}\frac{18i}{n^2} \\
&= \lim_{n\to\infty}-\frac{18}{n^2}\sum_{i=1}^{n}i \\
&= \lim_{n\to\infty}-\frac{18}{n^2}[\frac{n(n+1)}{2}] \\
&= \lim_{n\to\infty}-\frac{18n(n+1)}{2n^2} \\
&= -\lim_{n\to\infty}\frac{9(n+1)}{n} \\
&= -\lim_{n\to\infty}\frac{9n+9}{n} \\
&= -9
\end{aligned}$$

9. Evaluate the integral by interpreting it in terms of areas: $\int_{-1}^{2}(1-x)dx$
   Note that this is simply the area under the line $y=-x+1$ over $[-1,2]$ or–in other words–the area of a right triangle with base 3 and height 3. Thus, $\int_{-1}^{2}(1-x)dx=\frac{9}{2}$

10. Show that $\int_a^b xdx=\frac{b^2-a^2}{2}$
    Invoking problems 3 and 4,

$$
\begin{aligned}
\int_a^b x dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{b-a}{n}\ (a + \frac{(b-a)i}{n}) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{a(b-a)}{n} + \frac{b-a}{n} \sum_{i=1}^{n} \frac{(b-a)i}{n} \\
&= \lim_{n\to\infty} n[\frac{a(b-a)}{n}] + \frac{b-a}{n}[\frac{(b-a)}{n}] \sum_{i=1}^{n} i \\
&= \lim_{n\to\infty} a(b-a) + \frac{(b-a)^2}{n^2} \sum_{i=1}^{n} i \\
&= \lim_{n\to\infty} a(b-a) + \frac{(b-a)^2}{n^2}[\frac{n(n+1)}{2}] \\
&= \lim_{n\to\infty} a(b-a) + \frac{(n+1)(b-a)^2}{2n} \\
&= \lim_{n\to\infty} a(b-a) + \lim_{n\to\infty} \frac{(n+1)(b-a)^2}{2n} \\
&= a(b-a) + \lim_{n\to\infty} \frac{(b-a)^2 n + (b-a)^2}{2n} \\
&= a(b-a) + \frac{(b-a)^2}{2} \\
&= \frac{2a(b-a)}{2} + \frac{(b-a)^2}{2} \\
&= \frac{2ab - 2a^2 + b^2 - 2ab + a^2}{2} \\
&= \frac{b^2 - a^2}{2}
\end{aligned}
$$