Worksheet 27: Riemann Sums

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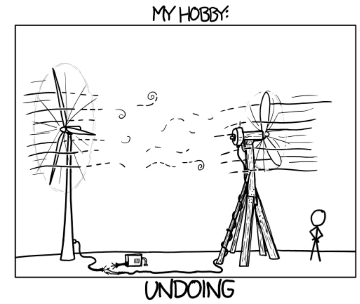

www.xkcd.com

1. Write a summation approximating the area under $f(x) = x^2$ over the interval $[-1, 1]$ with n rectangles where the height of each rectangle is given by the height at the right endpoint.

   Note that the interval has width 2, and so n rectangles gives a base of $\frac{2}{n}$ for each rectangle. Using the right endpoint of each rectangle to give us its height means the endpoints we need to plug into f are $-1 + \frac{2}{n}, -1 + \frac{2}{n} + \frac{2}{n}, \ldots, 1$ (if this isn't clear to you, draw the situation). Putting these together, we obtain the following summation:

   $$\sum_{i=1}^{n} \frac{2}{n}[f(-1 + \frac{2}{n}(i))] = \sum_{i=1}^{n} \frac{2}{n}[-1 + \frac{2}{n}(i)]^2$$

   Note that $-1 + \frac{2}{n}(i)$ from $i = 1$ to n gives precisely the end points above.

2. Write a summation approximating the area under $f(x) = x^2$ over the interval $[-1, 1]$ with n rectangles where the height of each rectangle is given by the height at the left endpoint.

   Note that the interval has width 2, and so n rectangles gives a base of $\frac{2}{n}$ for each rectangle. Using the left endpoint of each rectangle to give us its height means the endpoints we need to plug into f are $-1, -1 + \frac{2}{n}, -1 + \frac{2}{n} + \frac{2}{n}, \ldots, 1 - \frac{2}{n}$. Putting these together, we obtain the following summation:

   $$\sum_{i=0}^{n-1} \frac{2}{n}[f(-1 + \frac{2}{n}(i))] = \sum_{i=0}^{n-1} \frac{2}{n}[-1 + \frac{2}{n}(i)]^2$$

   Note that $-1 + \frac{2}{n}(i)$ from $i = 0$ to $n - 1$ gives precisely the end points above.

3. Write a summation approximating the area under $f(x) = x^2$ over the interval $[-1, 1]$ with n rectangles where the height of each rectangle is given by the maximum height over the rectangle's width (the upper sum).

   To be frank, we cheat slightly to do this in a nice way. In particular, we assume that the number of rectangles n is even, and so we may break the summation into the $\frac{n}{2}$ rectangles to the left of 0 and the $\frac{n}{2}$ rectangles to the right of 0. Doing this, note that the width of each rectangle is still unchanged at $\frac{n}{2}$ and that the maximum height over each rectangles base is simply its left endpoint if the rectangle is left of 0 and its right endpoint if the rectangle is right of 0. Thus,

   $$\sum_{i=0}^{\frac{n}{2}-1} \frac{2}{n}[f(-1 + \frac{2}{n}(i))] + \sum_{i=1}^{\frac{n}{2}} \frac{2}{n}[f(\frac{2}{n}(i))] = \sum_{i=0}^{\frac{n}{2}-1} \frac{2}{n}[-1 + \frac{2}{n}(i)]^2 + \sum_{i=1}^{\frac{n}{2}} \frac{2}{n}[\frac{2}{n}(i)]^2$$

4. Write a summation approximating the area under $f(x) = x^3$ over the interval $[-1, 1]$ with n rectangles (determine the height however you wish).

   Using right endpoints,

   $$\sum_{i=1}^{n} \frac{2}{n}[f(-1 + \frac{2}{n}(i))] = \sum_{i=1}^{n} \frac{2}{n}[-1 + \frac{2}{n}(i)]^3$$

5. (a) Plug in $n = 4$ in problem 4; what happens with the area for the first rectangle?

   We obtain:

   $$\frac{2}{4}[-1 + \frac{2}{4}]^3 + \frac{2}{4}[-1 + \frac{4}{4}]^3 + \frac{2}{4}[-1 + \frac{6}{4}]^3 + \frac{2}{4}[-1 + \frac{8}{4}]^3$$

   $$\frac{1}{2}[-\frac{1}{2}]^3 + \frac{1}{2}[0]^3 + \frac{1}{2}[\frac{1}{2}]^3 + \frac{1}{2}[1]^3$$

   $$\frac{1}{2}[1]^3$$

   $$\frac{1}{2}$$

   In particular, the area of the first rectangle is negative.

(b) Does this make sense? Why or why not?

Yes; we have to pick a base line somewhere. The x-axis is a natural choice, and so–if we drop below it–negative areas don't seem unreasonable (although, admittedly, a little disconnected from the real world).

6. If you wanted to determine the exact area for problems 1-4, what could you do mathematically?

   Take the limit as n goes to infinity.