Worksheet 26: Anti-Derivatives

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1. Find f

   (a) $f''(x) = 8x^3 + 5, f(1) = 0, f'(1) = 8$

   We simply start taking antiderivatives and solving for the constants:

$$f'(x) = 2x^4 + 5x + C$$

   By above, $f'(1) = 8$, so:

$$\begin{aligned} 2(1)^4 + 5(1) + C &= 8 \\ 7 + C &= 8 \\ C &= 1 \end{aligned}$$

   Thus,

$$f'(x) = 2x^4 + 5x + 1$$

   Then,

$$\begin{aligned} f(x) &= \frac{2}{5}x^5 + \frac{5}{2}x^2 + x + C \\ 0 &= \frac{2}{5}(1)^5 + \frac{5}{2}(1)^2 + 1 + C \\ -\frac{2}{5} - \frac{5}{2} - 1 &= C \\ C &= -\frac{39}{10} \end{aligned}$$

   And thus,

$$f(x) = \frac{2}{5}x^5 + \frac{5}{2}x^2 + x - \frac{39}{10}$$

   (b) $f''(t) = 2e^t + 3\sin(t), f(0) = 0, f'(\pi) = 0$

   We simply start taking antiderivatives and solving for the constants:

$$f'(x) = 2e^t - 3\cos(t) + C$$

   By above, $f'(\pi) = 0$, so:

$$\begin{aligned} 2e^\pi - 3\cos(\pi) + C &= 0 \\ 2e^\pi + 3 + C &= 0 \\ C &= -2e^\pi - 3 \end{aligned}$$

   Thus,

$$f'(x) = 2e^t - 3\cos(t) - 2e^\pi - 3$$

   Then,

$$\begin{aligned} f(x) &= 2e^t - 3\sin(t) - 2e^\pi x - 3x + C \\ 0 &= 2e^0 - 3\sin(0) - 2e^\pi(0) - 3(0) + C \\ 0 &= 2 + C \\ C &= -2 \end{aligned}$$

And thus,

$$f(x) = 2e^t - 3\sin(t) - 2e^{\pi}x - 3x - 2$$

(c) $f'''(x) = \cos(x), f(0) = 1, f'(0) = 2, f''(0) = 3$

We simply start taking antiderivatives and solving for the constants:

$$f''(x) = \sin(x) + C$$

By above, $f''(0) = 3$, so:

$$C = 3$$

Thus,

$$f''(x) = \sin(x) + 3$$

Then,

$$\begin{aligned} f'(x) &= -\cos(x) + 3x + C \\ 2 &= -\cos(0) + 3(0) + C \\ &= -1 + C \\ C &= 3 \end{aligned}$$

And thus,

$$f'(x) = -\cos(x) + 3x + 3$$

Finally,

$$\begin{aligned} f(x) &= -\sin(x) + \frac{3}{2}x^2 + 3x + C \\ 1 &= -\sin(0) + \frac{3}{2}(0)^2 + 3(0) + C \\ 1 &= C \end{aligned}$$

And so,

$$f(x) = -\sin(x) + \frac{3}{2}x^2 + 3x + 1$$

2. Find the anti-derivative:

(a) $f(x) = \frac{20x^4+12x}{4x^5+6x^2+1}$

$$F(x) = \ln(|4x^5 + 6x^2 + 1|)$$

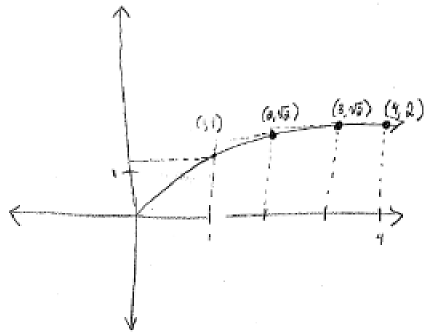
(b) $y = \frac{1}{1+x^2}$

$$F(x) = \arctan(x)$$

(c) $f(x) = -\sin(2x)e^{\cos(2x)}$

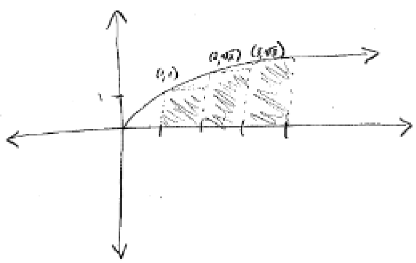
$$F(x) = \frac{1}{2}e^{\cos(2x)}$$

3. (a) Estimate the area under the graph of $f(x) = \sqrt{x}$ from $x = 0$ to $x = 4$ using four approximating rectangles and right endpoints. Sketch the graph and rectangles. Is your estimate an under or overestimate?



The estimate is $(1)(1) + (1)(\sqrt{2}) + (1)(\sqrt{3}) + (1)(2) = 3 + \sqrt{2} + \sqrt{3}$ which is an overestimate (note that all the rectangles are larger than the actual area under the curve.

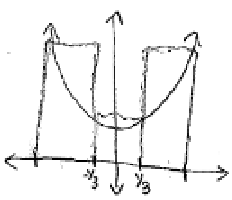
(b) Repeat part (a) with left endpoints.



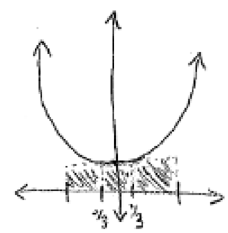
The estimate is $(1)(0) + (1)(1) + (1)(\sqrt{2}) + (1)(\sqrt{3}) = 1 + \sqrt{2} + \sqrt{3}$ which is an underestimate (note that all the rectangles are smaller than the actual area under the curve.

4. Evaluate the upper and lower sums for $f(x) = 1 + x^2$, $-1 \le x \le 1$ with $n = 3$ and $n = 4$. Sketch each.

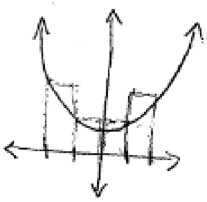
$\frac{2}{3}((-1)^2 + 1) + \frac{2}{3}((\frac{1}{3})^2 + 1) + \frac{2}{3}((1)^2 + 1) = \frac{92}{27}$



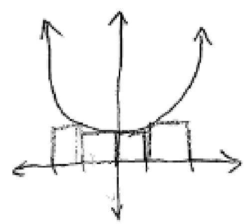
$\frac{2}{3}((-\frac{1}{3})^2 + 1) + \frac{2}{3}((0)^2 + 1) + \frac{2}{3}((\frac{1}{3})^2 + 1) = \frac{58}{27}$



$\frac{1}{2}((-1)^2 + 1) + \frac{1}{2}((-\frac{1}{2})^2 + 1) + \frac{1}{2}((\frac{1}{2})^2 + 1) + \frac{1}{2}((1)^2 + 1) = \frac{13}{4}$



$\frac{1}{2}((-\frac{1}{2})^2 + 1) + \frac{1}{2}((0)^2 + 1) + \frac{1}{2}((0)^2 + 1) + \frac{1}{2}((\frac{1}{2})^2 + 1) = \frac{9}{4}$



5. Determine a region whose area is equal to the limit $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}(5 + \frac{2i}{n})^{10}$.

Noting that $\frac{2}{n}$ has the expected form for an expression representing the width of each rectangle over an interval of size 2, and that this same expression is iterated through $(5 + x)^{10}$ in place of x (starting at $\frac{2}{n}$), we have both a function and an interval; thus,