Worksheet 24: Optimization

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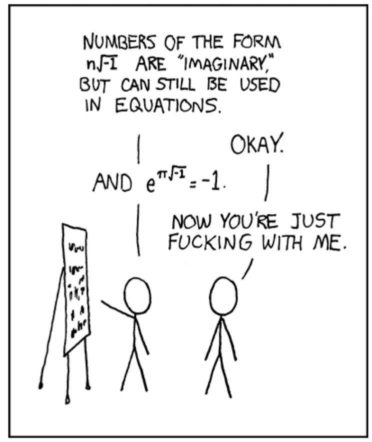


www.xkcd.com

1. Let $P = \frac{100I}{I^2+I+4}$. For what values of I is P maximum?

$$P = \frac{100I}{I^2+I+4}$$

Taking the derivative,

$$\begin{aligned} P' &= \frac{(I^2+I+4)(100) - 100I(2I+1)}{(I^2+I+4)^2} \\ &= \frac{(100)(I^2+I+4-2I^2-I)}{(I^2+I+4)^2} \\ &= \frac{(100)(-I^2+4)}{(I^2+I+4)^2} \\ &= \frac{(100)(-I+2)(I+2)}{(I^2+I+4)^2} \end{aligned}$$

Setting the top equal to 0,

$$\begin{aligned} 0 &= (-I+2)(I+2) \\ I &= 2, -2 \end{aligned}$$

Note further that the derivative is undefined whenever the bottom of the fraction is zero–but this only occurs when the bottom of the original function is also zero, and so these points are not in the domain of the function. We need only compare $I = 2, -2$:

$$\begin{aligned} P(2) &= 20 \\ P(-2) &= \frac{-100}{3} \end{aligned}$$

And thus $I = 2$ is the unique maximum.

2. Find the point on the curve $y = \sqrt{x}$ that is closest to the point $(3, 0)$.

Note that we wish to minimize the distance between the given function and the given point; we use, therefore, the distance formula:

$$d = \sqrt{(x-3)^2 + (y-0)^2}$$

Furthermore, since we only consider points on the function $y = \sqrt{x}$,

$$d = \sqrt{(x-3)^2 + (\sqrt{x}-0)^2}$$

Noting that d must be positive, and so maximizing d is the same as maximizing d^2, we consider

$$\begin{aligned} d^2 &= (x-3)^2 + (\sqrt{x}-0)^2 \\ &= (x-3)^2 + x \end{aligned}$$

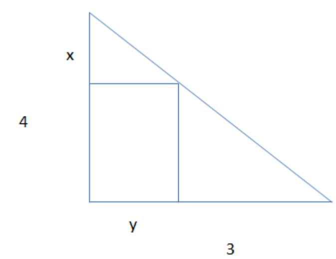
Taking the derivative,

$$\begin{aligned} \frac{d}{dx}[d^2] &= 2(x-3) + 1 \\ &= 2x - 5 \end{aligned}$$

Setting the derivative equal to zero,

$$\begin{aligned} 2x - 5 &= 0 \\ x &= \frac{5}{2} \\ y &= \sqrt{\frac{5}{2}} \end{aligned}$$

3. Find the area of the largest rectangle that can be inscribed in a right triangle with legs of lengths 3 cm and 4 cm if two sides of the rectangle lie along the legs.



Note first that the formula we would like to maximize is $A = (4 - x)(y)$. It remains, then, to eliminate either x or y from the equation (we need x in terms of y or vice versa). To do so, note that the large triangle above is similar to the triangle above the rectangle (they are both right and share an angle, and thus have the same angle measures for all angles). We therefore have,

$$\begin{aligned} \frac{x}{y} &= \frac{4}{3} \\ 4y &= 3x \\ y &= \frac{3}{4}x \end{aligned}$$

Substituting into our area equation,

$$\begin{aligned} A &= (4 - x)(\frac{3}{4}x) \\ &= 3x - \frac{3}{4}x^2 \end{aligned}$$

Taking the derivative,

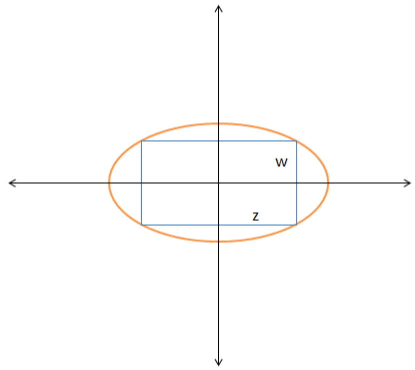
$$A' = 3 - \frac{3}{2}x$$

and setting it equal to 0,

$$\begin{aligned} 3 - \frac{3}{2}x &= 0 \\ x &= 2 \end{aligned}$$

And so the requested area is 3

4. ($\star$) Find the area of the largest rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.



The equation we wish to maximize is $A = (2w)(2z)$. Note first that the ellipse above has width a along the x-axis and height b along the y-axis. To eliminate either w or z, we solve the equation of the ellipse for its positive component:

$$\begin{aligned}
\frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1 \\
y &= \sqrt{(b^2)(1 - \frac{x^2}{a^2})} \\
&= \sqrt{b^2 - \frac{b^2}{a^2}x^2}
\end{aligned}$$

Note next that any acceptable w value will be equal to y in the equation above for some x. Similarly, note that the x value will be the same as z! We have, then,

$$w = \sqrt{b^2 - \frac{b^2}{a^2}z^2}$$

And so, substituting into the area equation,

$$A = 4z\sqrt{b^2 - \frac{b^2}{a^2}z^2}$$

Taking the derivative,

$$\begin{aligned}
A' &= 4\sqrt{b^2 - \frac{b^2}{a^2}z^2} + (4z)\frac{-\frac{b^2}{a^2}z}{\sqrt{b^2 - \frac{b^2}{a^2}z^2}} \\
&= \frac{4(b^2 - \frac{b^2}{a^2}z^2)}{\sqrt{b^2 - \frac{b^2}{a^2}z^2}} + \frac{-\frac{4b^2}{a^2}z^2}{\sqrt{b^2 - \frac{b^2}{a^2}z^2}} \\
&= \frac{4(b^2 - \frac{b^2}{a^2}z^2) - \frac{4b^2}{a^2}z^2}{\sqrt{b^2 - \frac{b^2}{a^2}z^2}} \\
&= \frac{b^2 - \frac{8b^2}{a^2}z^2}{\sqrt{b^2 - \frac{b^2}{a^2}z^2}}
\end{aligned}$$

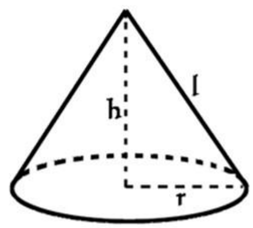
and setting it equal to 0,

$$\begin{aligned}
b^2 - \frac{8b^2}{a^2}z^2 &= 0 \\
z &= \sqrt{\frac{a^2}{8}} \\
&= \frac{a}{2\sqrt{2}}
\end{aligned}$$

Noting that making the bottom of the derivative equal to 0 is the same as making w 0 and is thus not a maximum, the requested area is:

$$\begin{aligned} A &= 4(\frac{a}{2\sqrt{2}})\sqrt{b^2 - \frac{b^2}{a^2}(\frac{a}{2\sqrt{2}})^2} \\ &= 4(\frac{a}{2\sqrt{2}})\sqrt{\frac{7}{8}b^2} \\ &= \frac{\sqrt{7}ab}{2} \end{aligned}$$

5. A cone-shaped drinking cup is to hold $27cm^3$ of water. Find the height and radius of the cup that will use that smallest amount of paper.



Noting that the surface area of a cone is given by

$$SA = \pi r^2 + \pi r l$$

We wish to minimize:

$$P = \pi r l$$

Furthermore, we are given that the volume of the cone must be $27cm^3$:

$$\begin{aligned} V &= 27 \\ \frac{1}{3}\pi r^2 h &= 27 \\ h &= \frac{81}{\pi r^2} \end{aligned}$$

Analyzing the diagram above, note that–by the Pythagorean theorem, $l = \sqrt{h^2 + r^2}$, and so:

$$l = \sqrt{(\frac{81}{\pi r^2})^2 + r^2}$$

Substituting into P,

$$\begin{aligned} P &= \pi r \sqrt{(\frac{81}{\pi r^2})^2 + r^2} \\ &= \pi r \sqrt{\frac{81^2}{\pi^2 r^4} + r^2} \end{aligned}$$

Taking the derivative,

$$\begin{aligned} P' &= \pi \sqrt{\frac{81^2}{\pi^2 r^4} + r^2} + (\pi r)\frac{\frac{(-4)81^2}{\pi^2} r^{-5} + 2r}{2\sqrt{\frac{81^2}{\pi^2 r^4} + r^2}} \\ &= \frac{\pi(\frac{81^2}{\pi^2 r^4} + r^2)}{\sqrt{\frac{81^2}{\pi^2 r^4} + r^2}} + \frac{\frac{(-2)81^2}{\pi} r^{-4} + \pi r^2}{\sqrt{\frac{81^2}{\pi^2 r^4} + r^2}} \\ &= \frac{\frac{81^2}{\pi r^4} + \pi r^2 + \frac{(-2)81^2}{\pi} r^{-4} + \pi r^2}{\sqrt{\frac{81^2}{\pi^2 r^4} + r^2}} \\ &= \frac{-\frac{81^2}{\pi} r^{-4} + 2\pi r^2}{\sqrt{\frac{81^2}{\pi^2 r^4} + r^2}} \end{aligned}$$

Setting the top equal to 0,

$$
\begin{aligned}
-\frac{81^2}{\pi}r^{-4} + 2\pi r^2 &= 0 \\
-\frac{81^2}{\pi r^4} + \frac{2\pi^2 r^6}{\pi r^4} &=
\end{aligned}
$$

and so,

$$
\begin{aligned}
-81^2 + 2\pi^2 r^6 &= 0 \\
r &= \sqrt[6]{\frac{3^8}{2\pi^2}} \\
h &= \frac{81}{\pi(\sqrt[6]{\frac{3^8}{2\pi^2}})^2} \\
&= \frac{3^4}{\pi(\sqrt[4]{\frac{3^8}{2\pi^2}})} \\
&= \frac{3^4}{\frac{3^2\sqrt{\pi}}{\sqrt[4]{2}}} \\
&= \frac{3^2\sqrt[4]{2}}{\sqrt{\pi}}
\end{aligned}
$$

Note that no undefined values for the derivative are acceptable since they give $l = 0$ and $r = 0$ respectively.

6. At which points on the curve $y = 1 + 40x^3 - 3x^5$ does the tangent line have the largest slope?

We wish to maximize $y' = 120x^2 - 15x^4$. Taking the derivative,

$$y'' = 240x - 60x^3$$

Setting it equal to zero,

$$
\begin{aligned}
240x - 60x^3 &= 0 \\
x(240 - 60x^2) &= 0 \\
240 - 60x^2 &= 0 \\
x^2 &= 4
\end{aligned}
$$

And thus,

$$x = 0, \pm 2$$

Comparing these points,

$$
\begin{aligned}
f'(0) &= 0 \\
f'(-2) = f(2) &= 240
\end{aligned}
$$

The points which maximizes the tangent line slope are therefore $(2, 225)$ and $(-2, -223)$.

7. (a) Show that if the profit $P(x)$ is a maximum, then the marginal revenue equals the marginal cost.
Assuming both marginal revenue (MR) and marginal cost (MC) exist, these are interpreted as the derivative of revenue (R) and cost (C) respectively. The profit function is $P(x) = R - C$, and so–assuming no undefined derivative values–$P(x) = 0$ when MC or C' is equal to MR or R'–the maximum by the extreme value theorem (assuming, again, that the derivative is defined on the entirety of the original domain).

(b) If $c(x) = 16,000 + 500x - 1.6x^2 + .004x^3$ is the cost function and $p(x) = 1700 - 7x$ is the demand function, find the production level that will maximize profit.

Note that the revenue function is given by $xp(x)$–that is, the price $p(x)$ times the number of items x. Thus,

$$R(x) = 1700x - 7x^2$$

Finding MC and MR,

$$\begin{aligned} MR &= 1700 - 14x \\ MC &= 500 - 3.2x + .012x^2 \end{aligned}$$

Setting them equal,

$$\begin{aligned} 1700 - 14x &= 500 - 3.2x + .012x^2 \\ 0 &= -1200 + 10.8x + .012x^2 \\ &= \frac{-1200000}{1000} + \frac{10800}{1000}x + \frac{12}{1000}x^2 \\ &= -1200000 + 10800x + 12x^2 \\ &= 12(x - 100)(x + 1000) \end{aligned}$$

And thus,

$$x = 100$$